

STACK OF S_3 -COVERS

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ABSTRACT. The aim of this paper is to study the geometry of the stack of S_3 -covers. We show that it has two irreducible components \mathcal{Z}_{S_3} and \mathcal{Z}_2 meeting in a “degenerate” point $\{0\}$, $\mathcal{Z}_2 - \{0\} \simeq \mathrm{BGL}_2$, while $(\mathcal{Z}_{S_3} - \{0\})$, which contains BS_3 as open substack, is a smooth and universally closed algebraic stack. More precisely we show that $\mathcal{Z}_{S_3} - \{0\} \simeq [X/\mathrm{GL}_2]$, where X is an explicit smooth non degenerate projective surface inside \mathbb{P}^7 intersection of five quadrics.

All these results are based on the description of certain families of S_3 -covers in terms of “building data”.

INTRODUCTION

Covers, ramified or unramified, with or without a group action, appear in many context of algebraic geometry (as well as other branches of mathematics) and they are often used to construct new varieties as total space of covers of a given variety. It is therefore useful to have a “recipe” for building up covers from the geometry of the base variety, in terms of divisors or vector bundles. The easiest example is the correspondence between double covers and line bundles together with a section of its second tensor power (provided the characteristic is not 2). Other classical results are [Mir85], [Par89] about triple covers and [Par91] about abelian covers. We also recall: [CE96, Cas96] about Gorenstein covers, [Eas11] about S_3 -covers as in the present paper, [Tok94, Rei99, CP17] about dihedral covers, [Tok02] about S_4 and A_4 -covers, [HM99] about quadruple covers, [AP12] about non-normal abelian covers, [Ahl20] about stacky covers, [BB17] about tamely ramified covers.

Besides constructing varieties, covers have been used to define some moduli spaces, especially moduli of covers of curves or surfaces, for example Hurwitz spaces (see e.g. [Ber13]). In this context we recall also [Pag16, AV04, BV12, PTT15]. In [Ton13a] I introduced the moduli stack $G\text{-Cov}$ of G -covers for a finite, flat and finitely presented group scheme G over some base S . Although $G\text{-Cov}$ and the various moduli of covers of curves/surfaces are both stacks parametrizing covers, we want to stress that they are very different objects: the geometric points in the first case are particular finite schemes (covers of a point) while in the second case are covers of curves/surfaces.

The problem of studying the geometry of $G\text{-Cov}$ is strictly linked with the problem of finding a “recipe” or, using the language of [Par91], a “building data” for constructing G -covers, with the difference that we need to describe covers of *any* scheme, with no geometric restriction on source or target (“classically” the base scheme is often assumed to be integral).

The stack $G\text{-Cov}$ is algebraic and finitely presented over S and contains BG , the stack of G -torsors, as an open substack. In particular it has a special component \mathcal{Z}_G , called the *main irreducible component*, which is the schematic closure of $\mathrm{BG} \subseteq G\text{-Cov}$. In [Ton13a] I investigated the geometry of those stacks in the abelian case, more precisely when G is a diagonalizable group scheme over $S = \mathrm{Spec} \mathbb{Z}$: it turns out that, except some cases in small ranks, this geometry can

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be very “wild” as one may expect from an Hilbert scheme. Nevertheless one can restrict the study to certain loci of G -Cov and, for instance, describe the smooth locus of \mathcal{Z}_G .

Moving to the *non-abelian* case the situation worsen, because there are no simple description of G -covers and therefore no obvious way to study the geometry of G -Cov. In [Ton17a] I propose an alternative interpretation of those covers as (non necessarily strong) monoidal functor, leading to some information about the geometry of G -Cov, for instance its reducibility for linearly reductive non-abelian groups G .

For simplicity let k be an algebraically closed field and assume that G is linearly reductive over k . A G -cover $f: X \rightarrow Y$ is completely determined by a collection of vector bundles (determining the module $f_*\mathcal{O}_X$), one for each irreducible representation of G and with equal rank, and a collection of maps between tensor products of those bundles (determining the ring structure of $f_*\mathcal{O}_X$). This data is very simple to describe, but it has to satisfy certain compatibility conditions (corresponding to the commutativity and associativity of $f_*\mathcal{O}_X$), which are expressed as commutative diagrams of maps between vector bundles. The complexity of the non-abelian case lies in the complexity and numerousness of those diagrams.

In this paper we consider the simplest non-abelian group $G = S_3$ for $\text{char } k \neq 2, 3$. In this case the complexity we discussed can be handled directly by listing all conditions and making sense of them. This lead to a “concrete” set of data describing S_3 -covers: a line bundle \mathcal{L} , a rank 2 vector bundle \mathcal{F} and maps $\alpha: \mathcal{L} \otimes \mathcal{F} \rightarrow \mathcal{F}$, $\beta: \text{Sym}^2 \mathcal{F} \rightarrow \mathcal{F}$ and $\langle -, - \rangle: \det \mathcal{F} \rightarrow \mathcal{L}$ making 5 diagrams commutes (see 2.9). From this description one deduce that $S_3\text{-Cov} \simeq [U/(\mathbb{G}_m \times \text{GL}_2)]$ for an explicit closed subscheme U of \mathbb{A}_k^{11} (see 3.1). We summarize the results obtained about the geometry of $S_3\text{-Cov}$ in the following:

Theorem. (3.5, 3.15, 4.20 and 4.21) *The stack $S_3\text{-Cov}$ has two irreducible components \mathcal{Z}_{S_3} and \mathcal{Z}_2 and they meet in a point $0 \in S_3\text{-Cov}$ (which corresponds to the “degenerate” S_3 -cover). Moreover the stack $S_3\text{-Cov} - \{0\}$ (resp. $\mathcal{Z}_{S_3} - \{0\}$) is the smooth locus of $S_3\text{-Cov}$ (resp. \mathcal{Z}_{S_3}), while $\mathcal{Z}_2 \simeq [\mathbb{A}^1/\mathbb{G}_m] \times \text{BGL}_2$. Finally $\mathcal{Z}_{S_3} - \{0\}$ is universally closed, more precisely*

$$\mathcal{Z}_{S_3} - \{0\} \simeq [X/\text{GL}_2]$$

for an explicit non degenerate smooth projective surface $X \subseteq \mathbb{P}_k^7$ complete intersection of five quadrics.

The above result is obtained by studying the equations of $U \subseteq \mathbb{A}^{11}$, describing certain loci of $S_3\text{-Cov}$ and mixing these points of view. The number of equations necessary to define $U \subseteq \mathbb{A}^{11}$ is proof of the complexity of the data associated with S_3 -covers and it suggests that the study of G -covers for general groups G needs an alternative approach (see 2.15).

We describe several open substacks of $S_3\text{-Cov}$ (and of \mathcal{Z}_{S_3}), namely the locus \mathcal{U}_ω where $\langle -, - \rangle$ is an isomorphism, which turns out to be equivalent to the stack of triple covers (see 4.3), the locus \mathcal{U}_α where α is nowhere a multiple of the identity (see 4.9) and the locus \mathcal{U}_β where β is nowhere zero (see 4.13). We show that $\mathcal{Z}_{S_3} - \{0\} = \mathcal{U}_\omega \cup \mathcal{U}_\alpha \cup \mathcal{U}_\beta$ and it is the smooth locus of \mathcal{Z}_{S_3} (see 4.20). For each of those open substacks and also for \mathcal{Z}_{S_3} itself we determine simpler sets of “data” for describing its S_3 -covers (i.e. its objects).

As mentioned before, in [Eas11] the author develops a similar theory of S_3 -covers, describing, locally and globally, the algebra defining S_3 -covers. The data provided coincides with the one we associate with covers in \mathcal{Z}_{S_3} . The present paper recovers the results of [Eas11] and expands them in several directions, by studying the geometry of $S_3\text{-Cov}$ and by describing several families of S_3 -covers.

For simplicity, in this introduction, we assumed to work over an algebraically closed field, but all results actually hold in general over $\mathbb{Z}[1/6]$. Moreover, instead of looking directly at the group S_3 , we work with $G = \mu_3 \rtimes \mathbb{Z}/2\mathbb{Z}$ over the ring $\mathbb{Z}[1/2]$. This is because the group G has a simpler representation theory, which simplifies the description of G -covers. Over $\mathbb{Z}[1/6]$ the groups G

and S_3 are isomorphic only étale locally, nevertheless, via the theory of bitorsors discussed in the Appendix, we show that there is an equivalence $G\text{-Cov} \simeq S_3\text{-Cov}$ inducing $\mathcal{Z}_G \simeq \mathcal{Z}_{S_3}$ and $BG \simeq BS_3$ (see 1.5).

This paper follows ideas from the last chapter of my Ph.D. thesis [Ton13b], but it introduces some improvements, as the study of the projective surface covering $\mathcal{Z}_{S_3} - \{0\}$. In [Ton13b] it is also present a characterization of S_3 -covers between regular schemes and some applications to surfaces. We plan to discuss and strengthen these results in a subsequent paper, applying also criteria from [Ton17a] and [Ton17b].

This paper is divided as follows. In the first section we apply the theory of bitorsors to the theory of G -covers. The second section defines the global “building data” for S_3 -covers, while the third one studies the geometry of $S_3\text{-Cov}$. The fourth and last section focuses instead on the geometry of \mathcal{Z}_{S_3} . There are two appendices, the first one about the theory of bitorsors, the second one about some general results on vector bundles.

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NOTATION

We denote by the letter T a scheme (over the given base if this is specified). It will be used as base for various algebro-geometric objects.

By a locally free sheaf we mean a locally free sheaf of finite rank.

A cover of T is a finite, flat and finitely presented morphism $f: X \rightarrow T$. This is the same as an affine map $f: X \rightarrow T$ such that $f_*\mathcal{O}_X$ is a locally free sheaf.

If G is a group scheme over a base S we denote by $\text{Loc}^G T$ (resp. $\text{QCoh}^G T$) the category of locally free sheaves (resp. quasi-coherent sheaves) over T together with an action of G . When $\pi: G \rightarrow S$ is affine, such an action is equivalent to a coaction of the sheaf of Hopf algebras $\pi_*\mathcal{O}_G$.

If \mathcal{F} is a locally free sheaf over T with a given basis $v_1, \dots, v_n \in \mathcal{F}$ we denote by $v_1^*, \dots, v_n^* \in \mathcal{F}^\vee$ its dual basis.

1. BITORSORS AND G -COVERS

In this section S is a base scheme and $G \rightarrow S$ is a finite, flat and finitely presented group scheme. We recall various definitions and properties about G -covers.

Definition 1.1. [Ton13a, Def. 2.1][Ton17a, Def 1.2] A G -cover over a scheme T is a cover $f: X \rightarrow T$ together with an action of G on X over T such that $f_*\mathcal{O}_X$ is fppf locally isomorphic to the regular representation $\mathcal{O}_T[G]$ as quasi-coherent sheaves with an action of G (but not necessarily preserving the ring structure).

We denote by $G\text{-Cov}$ the stack of G -covers over S .

Remark 1.2. [Ton13a, Prop 2.2, Thm 2.10] The stack $G\text{-Cov}$ is algebraic and finitely presented over S and contains BG as an open substack.

Definition 1.3. [Ton17a, Def 3.5] The stack \mathcal{Z}_G is the schematic closure of the open immersion $BG \rightarrow G\text{-Cov}$ and it is called the main irreducible component of $G\text{-Cov}$.

We apply now the theory of bitorsors (see Appendix A) to the theory of Galois covers.

Theorem 1.4. *Let G and H be flat, finite and finitely presented group schemes over a base scheme S . If P is an fpqc (G, H) -bitorsor over S then the functor Λ_P of A.8 fits in a commutative diagram*

$$\begin{array}{ccccccc} \mathrm{B}G & \longrightarrow & \mathcal{Z}_G & \longrightarrow & G\text{-Cov} & \longrightarrow & \mathrm{Sh}_{\mathrm{Sch}/S}^G & X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \Lambda_P & \downarrow \\ \mathrm{B}H & \longrightarrow & \mathcal{Z}_H & \longrightarrow & H\text{-Cov} & \longrightarrow & \mathrm{Sh}_{\mathrm{Sch}/S}^H & \frac{X \times P}{G} \end{array}$$

where the vertical arrows are equivalences. Moreover if Y is an S -scheme and $p \in P(Y)$ the induced composition $G_Y \rightarrow P_Y \rightarrow H_Y$ is an isomorphism of groups and $X \rightarrow \Lambda_P(X)$ is a natural isomorphism, equivariant with respect to $G_Y \rightarrow H_Y$, for $X \in \mathrm{Sh}_{\mathrm{Sch}/Y}^G$.

Proof. The functor $\Lambda_P: \mathrm{Sh}_{\mathrm{Sch}/S}^G \rightarrow \mathrm{Sh}_{\mathrm{Sch}/S}^H$ is an equivalence thanks to A.8. Taking into account A.5, it is enough to show that $\Lambda_P(X)$ is an H -cover if and only if X is a G -cover. Indeed any equivalence $G\text{-Cov} \rightarrow H\text{-Cov}$ restricting to an equivalence $\mathrm{B}G \rightarrow \mathrm{B}H$ has to induce an equivalence $\mathcal{Z}_G \rightarrow \mathcal{Z}_H$ of their schematic closures. Since being a G -cover or H -cover is a fppf local property of G -sheaves and fpqc G -torsors are fppf locally trivial, we can assume that P has a global section. The result then follows from A.5. \square

Theorem 1.5. *Let $G = \mu_n \times (\mathbb{Z}/n\mathbb{Z})^*$ and $H = \mathbb{Z}/n\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})^*$ for $n \geq 3$. The scheme*

$$P = \mu_n \times \mu_n^* = \mathrm{Spec}(A_P) \rightarrow \mathrm{Spec} \mathbb{Z}[1/n] \text{ where } A_P = \frac{\mathbb{Z}[1/n][x, y]}{(x^n - 1, \Phi_n(y))}$$

$\mu_n^* \subseteq \mu_n$ is the open and closed subscheme of primitive n -th roots and Φ_n is the cyclotomic polynomial of degree n , is a (G, H) -bitorsor with biaction

$$G \times P \times H \rightarrow P, (\zeta, l) \cdot (x, y) \cdot (i, m) = (\zeta x^l y^{li}, y^{lm})$$

In particular the functor Λ_P of A.8 induces equivalences as in 1.4 for $S = \mathrm{Spec} \mathbb{Z}[1/n]$. Moreover there is a canonical isomorphism

$$X/(\mathbb{Z}/n\mathbb{Z})^* \simeq \Lambda(X)/(\mathbb{Z}/n\mathbb{Z})^* \text{ for } X \in \mathrm{Sh}_{\mathbb{Z}[1/n]}^G$$

For $X = \mathrm{Spec} \mathcal{A} \in G\text{-Cov}$ the H -cover $\Lambda_P(X)$ is the spectrum of the sub-algebra

$$\mathcal{B} = \left(\frac{\mathcal{A}[x, y]}{(x^n - 1, \Phi_n(y))} \right)^G \subseteq \frac{\mathcal{A}[x, y]}{(x^n - 1, \Phi_n(y))} = \mathcal{A} \otimes A_P$$

The (left) H -action on $\mathcal{A} \otimes A_P$ is trivial on \mathcal{A} and given by $(i, m)x = xy^i$, $(i, m)y = y^m$ on A_P . The (left) G -action on $\mathcal{A} \otimes A_P$ is the given one on \mathcal{A} , while on A_P is generated by the μ_n -action for which $\deg x = -1$, $\deg y = 0$, while $l \in (\mathbb{Z}/n\mathbb{Z})^*$ acts as $x \mapsto x^l$, $y \mapsto y^{l'}$ where $l' = l^{-1} \in (\mathbb{Z}/n\mathbb{Z})^*$.

If $Y = \mu_n^* = \mathrm{Spec}(\mathbb{Z}[1/n][w]/(\Phi_n(w)))$ the section $(1, w) \in P(Y)$ induces the group isomorphism $\phi: H_Y \rightarrow G_Y$, $(i, m) \mapsto (w^i, m)$ and, for $\mathcal{A} \in G_Y\text{-Cov}$, the map

$$\mathcal{B} \subseteq \mathcal{A} \otimes A_P \rightarrow \mathcal{A}, x \mapsto 1, y \mapsto w$$

is an isomorphism, equivariant with respect to $\phi: H_Y \rightarrow G_Y$.

Proof. We apply A.11. Taking into account that $\underline{\mathrm{Aut}}(\mu_n) = \underline{\mathrm{Aut}}(\mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})^*$ and $\mu_n, \mathbb{Z}/n\mathbb{Z}$ are étale locally isomorphic over $\mathbb{Z}[1/n]$, the scheme $P = \mu_n \times \underline{\mathrm{Iso}}(\mathbb{Z}/n\mathbb{Z}, \mu_n)$ is a (G, H) -bitorsor. Part of the statement follows directly from 1.4. The remaining part consists in giving a more precise description of P . We have that

$$\underline{\mathrm{Iso}}(\mathbb{Z}/n\mathbb{Z}, \mu_n) \rightarrow \underline{\mathrm{Hom}}(\mathbb{Z}/n\mathbb{Z}, \mu_n) = \mu_n$$

is the locus of $\omega \in \mu_n$ such that the induced map $\mathbb{Z}/n\mathbb{Z} \rightarrow \mu_n$ is an isomorphism. Just looking at the order of ω , we see that $\omega \in \mu_n - \mu_d$ for any $d \mid n$ with $d < n$. As $\mu_d \subseteq \mu_n$ is an étale subgroup, it is an open and closed subscheme. Therefore $\omega \in \mu_n^* = \mu_n - \cup_{d \mid n, d < n} \mu_d$, which is an open and closed subscheme. Moreover we also have $\mu_n^* = \text{Spec}(\mathbb{Z}[1/n][y]/(\Phi_n(y)))$ by definition of Φ_n . The condition $\omega \in \mu_n^*$ means that ω is a primitive n -th root in all the residue fields of its base. This easily implies the equality $\text{Iso}(\mathbb{Z}/n\mathbb{Z}, \mu_n) = \mu_n^*$. The description of P in the statement follows, while the biaction of G and H can be computed directly from the definition in A.11.

We are left with the second part of the statement, so let $X = \text{Spec } \mathcal{A}$ be a G -cover. By definition $\Lambda_P(X) = (X \times P)/G$, where G acts on the right: $(x, p)g = (xg, g^{-1}p)$. In particular $\Lambda_P(X)$ is the spectrum of $(\mathcal{A} \otimes A_P)^G \subseteq \mathcal{A} \otimes A_P$. Here the right G -action on P , $- \star g = g^{-1} -$ induces a left G -action on (the functor associated with) A_P : $\zeta \in \mu_n$ acts as $x \mapsto \zeta^{-1}x$, $y \mapsto y$, while $(1, l)$, since its inverse in G is $(1, l')$, acts as $x \mapsto x'$, $y \mapsto y'$. In particular, as μ_n -comodule, A_P satisfies $\deg x = -1$ and $\deg y = 0$ and the total G action on $\mathcal{A} \otimes A_P$ is the diagonal one, as claimed in the statement. The H -action on $\Lambda_P(X) = (X \times P)/G$ is non trivial only on P , from which we deduce the H -action on $\mathcal{A} \otimes A_P$ and its subalgebra.

The last part follows directly from the last part of Theorem 1.4. \square

2. DATA FOR $(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$ -COVERS OR S_3 -COVERS

In this section we work over the ring \mathcal{R} of integers with 2 inverted, that is $\mathcal{R} = \mathbb{Z}[1/2]$ and with the symbol G we will always denote the group scheme $G = \mu_3 \rtimes \mathbb{Z}/2\mathbb{Z}$ defined over \mathcal{R} , where the action of $\mathbb{Z}/2\mathbb{Z}$ on μ_3 is given by the inversion, that is $\mathbb{Z}/2\mathbb{Z} \simeq \text{Aut}(\mu_3)$. Note that, in this case, $\mu_2 \simeq \mathbb{Z}/2\mathbb{Z}$. We denote by $\sigma \in \mathbb{Z}/2\mathbb{Z}(\mathcal{R})$ the non trivial generator of $\mathbb{Z}/2\mathbb{Z}$. We will also think of σ as an element of $G(\mathcal{R})$.

2.1. The group $(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$ and its representation theory. The group G is a linearly reductive group over \mathcal{R} (see [AOV08, Prop 2.6, Thm 2.16]).

Set $V_0 = \mathcal{R}, V_1, V_2$ for the representations of μ_3 corresponding to its characters in $\mathbb{Z}/3\mathbb{Z}$. Moreover consider the set I_G of G -representations

$$\mathcal{R}, A = V_\chi, V = \text{ind}_{\mu_3}^G V_1$$

where $\chi: G \rightarrow \mathbb{G}_m$ is induced by the non trivial character of $\mathbb{Z}/2\mathbb{Z}$. Since 2 is invertible in \mathcal{R} , the representations in I_G restricts over \mathbb{Q} to the irreducible representations of $G \times \overline{\mathbb{Q}} \simeq S_3$. In other words we have proved that:

Proposition 2.1. *The pair (G, I_G) is a good linearly reductive group over \mathcal{R} in the sense of [Ton17a, Def 1.11].*

We setup the following notation and we will use it throughout the paper. We consider the following basis $1 \in \mathcal{R}, 1_A \in A$ and $v_1, v_2 \in V = V_1 \oplus V_2$ such that $v_i \in V_i$. Moreover since σ exchanges V_1 and V_2 , we also assume that $\sigma(v_1) = v_2, \sigma(v_2) = v_1$. Now we describe the tensor products of the representations in I_G . We have

$$A \otimes A \simeq \mathcal{R}, 1_A \otimes 1_A \longrightarrow 1 \text{ and } A \otimes V \simeq V, 1_A \otimes v_1 \longrightarrow -v_1, 1_A \otimes v_2 \longrightarrow v_2$$

and, if we set $v_{ij} = v_i \otimes v_j \in V \otimes V$,

$$\mathcal{R} \oplus A \oplus V \simeq V \otimes V, 1 \longrightarrow v_{12} + v_{21}, 1_A \longrightarrow v_{12} - v_{21}, v_1 \longrightarrow v_{22}, v_2 \longrightarrow v_{11}$$

Finally note that the G -equivariant projection $V \otimes V \rightarrow \mathcal{R}$, $v_{ij} \rightarrow 1 - \delta_{ij}$, where δ_{ij} is the Kronecker symbol, yields an isomorphism

$$V \simeq V^\vee, v_1 \longrightarrow v_2^*, v_2 \longrightarrow v_1^*$$

The above discussion allows us to conclude the following:

2.2. Global description of $(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$ -covers. In this section we want to describe the data needed to define a G -cover over any \mathcal{R} -scheme. We proceed in the following way. First we introduce such data, then we will state the precise relationship with G -covers and only after we will prove all the claims. We remark here that the global description obtained, although with a different notation, has already been introduced in [Eas11].

We define the stack \mathcal{Y} over \mathcal{R} whose objects over an \mathcal{R} -scheme T are sequences $\chi = (\mathcal{L}, \mathcal{F}, \alpha, \beta, \langle -, - \rangle)$ where: \mathcal{L} is an invertible sheaf, \mathcal{F} is a rank 2 locally free sheaf and $\alpha, \beta, \langle -, - \rangle$ are maps

$$\mathcal{L} \otimes \mathcal{F} \xrightarrow{\alpha} \mathcal{F}, \text{Sym}^2 \mathcal{F} \xrightarrow{\beta} \mathcal{F}, \det \mathcal{F} \xrightarrow{\langle -, - \rangle} \mathcal{L}$$

With an object $\chi \in \mathcal{Y}$ as above we associate the map $(-, -)_\chi: \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{O}_T$ given by

$$(2.1) \quad (-, -)_\chi: \mathcal{F} \otimes \mathcal{F} \simeq \mathcal{F}^\vee \otimes \det \mathcal{F} \otimes \mathcal{F} \xrightarrow{\text{id} \otimes \langle -, - \rangle \otimes \text{id}} \mathcal{F}^\vee \otimes \mathcal{L} \otimes \mathcal{F} \xrightarrow{\text{id} \otimes \alpha} \mathcal{F}^\vee \otimes \mathcal{F} \rightarrow \mathcal{O}_T$$

where we are using the canonical isomorphism $\mathcal{F} \simeq \mathcal{F}^\vee \otimes \det \mathcal{F}$. Notice that, although we are using the symbol $(-, -)$ of a symmetric product, $(-, -)_\chi$ is not necessarily symmetric. Moreover we also associate with χ the maps $\gamma_\chi, \gamma'_\chi: \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{O}_T \oplus \mathcal{L}$ given by

$$\gamma_\chi = (-, -)_\chi + \langle -, - \rangle, \quad \gamma'_\chi = (-, -)_\chi - \langle -, - \rangle$$

Finally we define (see B.2)

$$(2.2) \quad m_\chi = (1/2) \text{tr}(\mathcal{L}^2 \otimes \mathcal{F} \xrightarrow{\text{id}_{\mathcal{L}} \otimes \alpha} \mathcal{L} \otimes \mathcal{F} \xrightarrow{\alpha} \mathcal{F}): \mathcal{L}^2 \rightarrow \mathcal{O}_T$$

and

$$\mathcal{A}_\chi = \mathcal{O}_T \oplus \mathcal{L} \oplus \mathcal{F}_1 \oplus \mathcal{F}_2 \text{ with } \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}$$

For convenience we also set $\mathcal{L}_\chi = \mathcal{L}$, $\mathcal{F}_\chi = \mathcal{F}$, $\alpha_\chi = \alpha$, $\beta_\chi = \beta$ and $\langle -, - \rangle_\chi = \langle -, - \rangle$: given $\chi \in \mathcal{Y}$ we don't need to specify the whole sequence to refer to one of its elements, for instance we could simply write $\beta_\chi = 0$ and so on. On the other hand, when χ is given and there is no possibility of confusion, we will omit the $-\chi$ and simply write $(-, -), \gamma, \gamma', m, \mathcal{A}$ or $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, (-, -), \langle -, - \rangle) \in \mathcal{Y}$.

Definition 2.2. We denote by $\text{LRings}_{\mathcal{R}}^G$ the stack of locally free shaves \mathcal{A} of finite rank with a coaction of G and an equivariant multiplication map $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ (not necessarily commutative or associative) with an invariant unit $1 \in \mathcal{A}^G$.

Proposition 2.3. *Given $\chi \in \mathcal{Y}$ as above, the sheaf \mathcal{A}_χ has a unique G -comodule structure such that $\mathcal{F}_0 = \mathcal{O}_T \oplus \mathcal{L}, \mathcal{F}_1, \mathcal{F}_2$ define the μ_3 -action and σ acts as $-\text{id}_{\mathcal{L}}$ on \mathcal{L} and induces $\text{id}_{\mathcal{F}}: \mathcal{F}_1 \rightarrow \mathcal{F}_2$.*

This proposition will be proved later. We makes \mathcal{A}_χ into an object of $\text{LRings}_{\mathcal{R}}^G$ defining the following multiplication map $\mathcal{A}_\chi \otimes \mathcal{A}_\chi \rightarrow \mathcal{A}_\chi$:

$$(2.3) \quad \begin{aligned} \mathcal{L}^2 &\xrightarrow{m} \mathcal{O}_T, \quad \mathcal{L} \otimes \mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F}_1, \quad \mathcal{L} \otimes \mathcal{F}_2 \xrightarrow{-\alpha} \mathcal{F}_2, \quad \mathcal{F}_1 \otimes \mathcal{L} \xrightarrow{\hat{\alpha}} \mathcal{F}_1, \quad \mathcal{F}_2 \otimes \mathcal{L} \xrightarrow{-\hat{\alpha}} \mathcal{F}_2 \\ \mathcal{F}_1 \otimes \mathcal{F}_1 &\xrightarrow{\beta} \mathcal{F}_2, \quad \mathcal{F}_2 \otimes \mathcal{F}_2 \xrightarrow{\beta} \mathcal{F}_1, \quad \mathcal{F}_1 \otimes \mathcal{F}_2 \xrightarrow{\eta_1 + \eta_2} \mathcal{O}_T \oplus \mathcal{L}, \quad \mathcal{F}_2 \otimes \mathcal{F}_1 \xrightarrow{\eta_1 - \eta_2} \mathcal{O}_T \oplus \mathcal{L} \end{aligned}$$

where $\hat{\alpha}$ is obtained by α just swapping the factors in the source and, for future reference, we set $\eta_1 = (-, -)$ and $\eta_2 = \langle -, - \rangle$. We are implicitly assuming that the maps $\mathcal{O}_T \otimes \mathcal{A}_\chi, \mathcal{A}_\chi \otimes \mathcal{O}_T \rightarrow \mathcal{A}_\chi$ are just the usual isomorphisms, or, in other words, that $1 \in \mathcal{O}_T$ is the unity for \mathcal{A}_χ .

We want now to give a list of equations involving the maps $\alpha, \beta, \langle -, - \rangle$, which we will show are the relationships needed for the associativity of \mathcal{A}_χ . Such equations will be 'local' relations and therefore we introduce the following notation:

Notation 2.4. When we fix a generator t of \mathcal{L} , the maps $m, \alpha, \beta, \langle -, - \rangle$ will be thought of as: $m \in \mathcal{O}_T$, given by $m(t \otimes t)$; $\alpha: \mathcal{F} \rightarrow \mathcal{F}$, given by “ $\alpha(u) = \alpha(t \otimes u)$ ”; $\langle -, - \rangle: \det \mathcal{F} \rightarrow \mathcal{O}_S$, given by “ $\langle u, v \rangle = \langle u, v \rangle t$ ”. When we will say that some particular relation among the maps $m, \alpha, \beta, \langle -, - \rangle$ locally holds, this will always mean that such relation holds as soon as basis t and y, z of, respectively, \mathcal{L} and \mathcal{F} are given.

For instance, the following relation holds locally (use B.1):

$$(2.4) \quad (u, v) = \langle \alpha(v), u \rangle \text{ for } u, v \in \mathcal{F}$$

Consider the equations:

$$(2.5) \quad \alpha^2 = \text{id}_{\mathcal{F}}$$

$$(2.6) \quad \langle \alpha(u), v \rangle = \langle \alpha(v), u \rangle \quad \text{for } u, v \in \mathcal{F}$$

$$(2.7) \quad \alpha(\beta(u \otimes v)) = -\beta(u \otimes \alpha(v)) \quad \text{for } u, v \in \mathcal{F}$$

$$(2.8) \quad \beta(\beta(u \otimes v) \otimes w) = \langle \alpha(w), v \rangle u + \langle v, w \rangle \alpha(u) \quad \text{for } u, v, w \in \mathcal{F}$$

$$(2.9) \quad \langle u, \beta(v \otimes w) \rangle = \langle w, \beta(u \otimes v) \rangle \quad \text{for } u, v, w \in \mathcal{F}$$

Theorem 2.5. *The map of stacks*

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \text{LRings}_{\mathcal{R}}^G \\ \chi = (\mathcal{L}, \mathcal{F}, \alpha, \beta, \langle -, - \rangle) & \longmapsto & \mathcal{A}_{\chi} \end{array}$$

is well defined, fully faithful and induces an equivalence between the substack of \mathcal{Y} of objects that locally satisfy the relations (2.5), (2.6), (2.7), (2.8), (2.9) and $G\text{-Cov}$ (where a cover is thought of as its corresponding sheaf of algebras).

Notation 2.6. Assuming Theorem 2.5, we will think of $G\text{-Cov}$ as substack of \mathcal{Y} , using for instance expressions like $\chi = (\mathcal{L}, \mathcal{F}, \alpha, \beta, \langle -, - \rangle) \in G\text{-Cov}$.

The aim of this section is to prove Theorem 2.5. We will often use results and notations from [Ton14] and [Ton17a].

Denote by \mathcal{X} the stack whose fibers over a scheme T is the groupoid of pseudo-monoidal (see [Ton14, Def 2.21]) and \mathcal{R} -linear functors $\Gamma: \text{Loc}^G \mathcal{R} \rightarrow \text{Loc} T$ such that $\text{rk} \Gamma_U = \text{rk} U$ for all $U \in \text{Loc}^G \mathcal{R}$, $\Gamma_{\mathcal{R}} = \mathcal{O}_T$ and $1 \in \Gamma_{\mathcal{R}}$ is a unity. We are going to embed \mathcal{Y} into \mathcal{X} .

We use results and notations from 2.1. By [Ton17a, Lemma 1.9] and [Ton17a, Rmk 1.17] there are isomorphisms

$$\bigoplus_{W \in I_G} \text{Hom}^G(W, U) \otimes W \rightarrow U \quad \text{and} \quad \bigoplus_{W \in I_G} \text{Hom}^G(W, U) \otimes \Gamma_W \rightarrow \Gamma_U$$

natural for $U \in \text{Loc}^G \mathcal{R}$ and for a \mathcal{R} -linear functor $\Gamma: \text{Loc}^G \mathcal{R} \rightarrow \text{Loc} T$. It follows that Γ is completely determined by the collection of locally free sheaves $(\Gamma_W)_{W \in I_G}$. Moreover a pseudo-monoidal structure on Γ corresponds to a sequence of maps

$$\Gamma_W \otimes \Gamma_{W'} \rightarrow \Gamma_{W \otimes W'} \xrightarrow{\cong} \bigoplus_{Z \in I_G} \text{Hom}^G(Z, W \otimes W') \otimes \Gamma_Z \text{ for } W, W' \in I_G$$

Since in 2.1 we fixed basis for the modules $\text{Hom}^G(Z, W \otimes W')$ as above, it follows that an object $\Gamma \in \mathcal{X}(T)$ can be represented by a sequence $(\mathcal{L}, \mathcal{F}, m, \alpha, \hat{\alpha}, \eta_1, \eta_2, \beta)$ where

$$\mathcal{L}(= \Gamma_A), \quad \mathcal{F}(= \Gamma_V)$$

are an invertible sheaf and a rank 2 sheaf on T respectively and

$$\mathcal{L} \otimes \mathcal{L} \xrightarrow{m} \mathcal{O}_T, \quad \mathcal{L} \otimes \mathcal{F} \xrightarrow{\alpha} \mathcal{F}, \quad \mathcal{F} \otimes \mathcal{L} \xrightarrow{\hat{\alpha}} \mathcal{F}, \quad \mathcal{F} \otimes \mathcal{F} \xrightarrow{\eta_1 \oplus \eta_2 \oplus \beta} \mathcal{O}_T \oplus \mathcal{L} \oplus \mathcal{F}$$

are maps. In particular \mathcal{Y} can be embedded in \mathcal{X} by sending $\chi = (\mathcal{L}, \mathcal{F}, \alpha, \beta, \langle -, - \rangle) \in \mathcal{Y}$ to the sequence $(\mathcal{L}, \mathcal{F}, m_\chi, \alpha, \hat{\alpha}, (-, -)_\chi, \langle -, - \rangle, \beta)$, where $\hat{\alpha}$ is obtained from α exchanging the factors in the source.

By [Ton17a, Thm A], [Ton17a, Rmk 1.17] and [Ton14, Thm 8.6] there is a fully faithful functor

$$\mathcal{B}_*: \mathcal{X} \rightarrow \text{LRings}_{\mathcal{R}}^G, \quad \mathcal{B}_\Gamma = \bigoplus_{W \in I_G} W^\vee \otimes \Gamma_W$$

which restricts to an equivalence between the substack of \mathcal{X} of monoidal functors and $G\text{-Cov}$. In particular a G -cover is (the spectrum of) a \mathcal{B}_Γ (for some $\Gamma \in \mathcal{X}$) which is commutative and associative. Here the multiplication of \mathcal{B}_Γ is induced by the pseudo-monoidal structure on Γ .

Let us assume that Γ is the functor associated with

$$\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \hat{\alpha}, \eta_1, \eta_2, \beta) \in \mathcal{X}(T)$$

In this case we simply write $\mathcal{B}_\chi = \mathcal{B}_\Gamma$. The choice of basis for the modules in I_G in 2.1 defines an isomorphism

$$\begin{aligned} \mathcal{A} &= \mathcal{O}_T \oplus \mathcal{L} \oplus (\mathcal{F}_1 \oplus \mathcal{F}_2) \rightarrow (\mathcal{R}^\vee \otimes \mathcal{O}_T) \oplus (A^\vee \otimes \mathcal{L}) \oplus (V^\vee \otimes \mathcal{F}) = \mathcal{B}_\chi \\ 1 \oplus x \oplus f_1 \oplus f_2 &\mapsto (1^* \otimes 1) \oplus (1_A^* \otimes x) \oplus [(v_2^* \otimes f_1) \oplus (v_1^* \otimes f_2)] \end{aligned}$$

Lemma 2.7. *The G -comodule structure and the multiplication induced on \mathcal{A} by \mathcal{B}_χ are the ones described in 2.3 and (2.3) respectively. In particular $(\mathcal{B}_*)|_{\mathcal{Y}}$ is the map \mathcal{A}_* of Theorem 2.5, which is well defined and fully faithful.*

Proof. The claim about the G -comodule structure is clear, so we just have to translate the multiplication. This is possible using properties 1) to 4) of [Ton17a, Rmk 1.17]. We are going to discuss in details only one case, while for the other ones we will just present the relevant computations.

The sheaf $\mathcal{O}_T \oplus \mathcal{L}$ is a $\mathbb{Z}/2\mathbb{Z}$ -cover and we claim that the induced multiplication $\mathcal{L} \otimes \mathcal{L} \rightarrow \mathcal{O}_T$ is just m . By [Ton17a, Rmk 1.17], in particular point 4), there is a commutative diagram

$$\begin{array}{ccccc} A^\vee \otimes \Gamma_A \otimes A^\vee \otimes \Gamma_A & \longrightarrow & \mathcal{B}_\chi \otimes \mathcal{B}_\chi & \longrightarrow & \mathcal{B}_\chi \\ \downarrow & & & & \uparrow \\ (A \otimes A)^\vee \otimes \Gamma_A \otimes \Gamma_A & \longrightarrow & (A \otimes A)^\vee \otimes \Gamma_{A \otimes A} & \xrightarrow{(\xi^\vee)^{-1} \otimes \Gamma_\xi} & \mathcal{R}^\vee \otimes \Gamma_{\mathcal{R}} \end{array}$$

where $\xi: A \otimes A \rightarrow \mathcal{R}$ is *any* G -equivariant isomorphism. On the other hand the composition $\mathcal{L} \otimes \mathcal{L} = \Gamma_A \otimes \Gamma_A \rightarrow \Gamma_{A \otimes A} \xrightarrow{\Gamma_\xi} \Gamma_{\mathcal{R}} = \mathcal{O}_T$ is m if and only if ξ is the isomorphism chosen in section 2.1, i.e. such that $\xi(1_A \otimes 1_A) = 1$, because this is the way we defined the correspondence $\chi \leftrightarrow \Gamma$. Since, in this case, $(\xi^\vee)^{-1}((1_A \otimes 1_A)^*) = 1^*$, by diagram chasing we see that the multiplication of \mathcal{B}_χ maps

$$(1_A^* \otimes x) \otimes (1_A^* \otimes y) \mapsto 1^* \otimes m(x \otimes y) \text{ for } x, y \in \mathcal{L} = \Gamma_A$$

This shows that the multiplication $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ induced by \mathcal{B}_χ restricts to $m: \mathcal{L} \otimes \mathcal{L} \rightarrow \mathcal{O}_T$ as claimed.

Since $(A \otimes V)^\vee \rightarrow V^\vee$ maps $(1_A \otimes v_1)^* \mapsto -v_1^*$, $(1_A \otimes v_2)^* \mapsto v_2^*$ the induced map $\mathcal{L} \otimes (\mathcal{F}_1 \oplus \mathcal{F}_2) \rightarrow \mathcal{F}_1 \oplus \mathcal{F}_2$ splits into α and $-\alpha$. Similarly, the maps $(\mathcal{F}_1 \oplus \mathcal{F}_2) \otimes \mathcal{L} \rightarrow \mathcal{F}_1 \oplus \mathcal{F}_2$ splits as $\hat{\alpha}$ and $-\hat{\alpha}$. Finally one has

$$(V \otimes V)^\vee \rightarrow \mathcal{R}^\vee \oplus A^\vee \oplus V^\vee, \quad v_{11}^* \mapsto v_2^*, \quad v_{22}^* \mapsto v_1^*, \quad v_{12}^* \mapsto 1^* + 1_A^*, \quad v_{21}^* \mapsto 1^* - 1_A^*$$

which implies that the induced multiplication on \mathcal{A} is given by

$$\mathcal{F}_1 \otimes \mathcal{F}_1 \xrightarrow{\beta} \mathcal{F}_2, \mathcal{F}_2 \otimes \mathcal{F}_2 \xrightarrow{\beta} \mathcal{F}_1, \mathcal{F}_1 \otimes \mathcal{F}_2 \xrightarrow{\eta_1 + \eta_2} \mathcal{O}_T \oplus \mathcal{L}, \mathcal{F}_2 \otimes \mathcal{F}_1 \xrightarrow{\eta_1 - \eta_2} \mathcal{O}_T \oplus \mathcal{L}$$

□

In order to prove Theorem 2.5 we have to show that \mathcal{A} is (the algebra of) a G -cover if and only if $\chi \in \mathcal{Y}$ and it satisfies the properties listed in Theorem 2.5. Thus we have to translate commutativity and associativity conditions of \mathcal{A} .

2.2.1. Commutativity conditions. We claim that \mathcal{A} is commutative if and only if: $\beta: \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F}$ is symmetric, $\hat{\alpha}: \mathcal{F} \otimes \mathcal{L} \rightarrow \mathcal{F}$ is obtained from $\alpha: \mathcal{L} \otimes \mathcal{F} \rightarrow \mathcal{F}$ swapping factors in the source, η_1 is symmetric, η_2 is antisymmetric. Indeed by [Ton14, Prop 2.25 and Prop 2.26] \mathcal{A} is commutative if and only if Γ is symmetric. Moreover this is equivalent to: for $U, U' \in I_G$ the maps $\Gamma_U \otimes \Gamma_{U'} \rightarrow \Gamma_{U \otimes U'}$ and $\Gamma_{U'} \otimes \Gamma_U \rightarrow \Gamma_{U' \otimes U} \xrightarrow{\Gamma_{\text{swap}}} \Gamma_{U \otimes U'}$ differ by a swap of factors in the source. For $U = U' = A$ we see that $m: \mathcal{L} \otimes \mathcal{L} \rightarrow \mathcal{O}_T$ is automatically symmetric because \mathcal{L} has rank 1. For $U = A, U' = V$ (or the converse) we obtain the relation between α and $\hat{\alpha}$. For $U = U' = V$, notice that the swap map on $V \otimes V \simeq \mathcal{R} \oplus A \oplus V$ is the identity on \mathcal{R} and V and minus the identity on A . This translates in the symmetry of η_1 and β and the antisymmetry of η_2 .

2.2.2. Associativity conditions. Let us assume that \mathcal{A} is commutative. Moreover we use the notation $\eta_1 = (-, -)$, $\eta_2 = \langle -, - \rangle$, $\gamma = \eta_1 + \eta_2$ and $\gamma' = \eta_1 - \eta_2$. We now express some diagrams that have to commute if \mathcal{A} is associative. We use the notation introduced in 2.4.

•

$$(2.10) \quad \begin{array}{ccc} \mathcal{L} \otimes \mathcal{L} \otimes \mathcal{F}_1 & \xrightarrow{m \otimes \text{id}} & \mathcal{O}_S \otimes \mathcal{F}_1 \\ \downarrow \text{id} \otimes \alpha & & \downarrow \text{id} \\ \mathcal{L} \otimes \mathcal{F}_1 & \xrightarrow{\alpha} & \mathcal{F}_1 \end{array}$$

Locally we obtain the condition (2.5).

•

$$(2.11) \quad \begin{array}{ccc} \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{L} & \xrightarrow{\gamma \otimes \text{id}} & (\mathcal{O}_S \oplus \mathcal{L}) \otimes \mathcal{L} \\ \downarrow \text{id} \otimes \hat{\alpha} & & \downarrow \text{id} \\ & & \mathcal{L} \otimes \mathcal{L} \oplus \mathcal{L} \\ & & \downarrow m \oplus \text{id} \\ \mathcal{F}_1 \otimes \mathcal{F}_2 & \xrightarrow{\gamma} & \mathcal{O}_S \oplus \mathcal{L} \end{array}$$

The commutativity of this diagram is locally equivalent to $(u, \alpha(v)) = -m\langle u, v \rangle$, $(u, v) = -\langle u, \alpha(v) \rangle$ and, assuming (2.5), to (2.4).

•

$$(2.12) \quad \begin{array}{ccc} \mathcal{F}_1 \otimes \mathcal{F}_1 \otimes \mathcal{L} & \xrightarrow{\beta \otimes \text{id}} & \mathcal{F}_2 \otimes \mathcal{L} \\ \downarrow \text{id} \otimes \hat{\alpha} & & \downarrow -\hat{\alpha} \\ \mathcal{F}_1 \otimes \mathcal{F}_1 & \xrightarrow{\beta} & \mathcal{F}_2 \end{array}$$

The commutativity of this diagram is locally equivalent to (2.7).

$$(2.13) \quad \begin{array}{ccc} \mathcal{F}_2 \otimes \mathcal{F}_2 \otimes \mathcal{F}_1 & \xrightarrow{\beta \otimes \text{id}} & \mathcal{F}_1 \otimes \mathcal{F}_1 \\ \downarrow \text{id} \otimes \gamma' & & \downarrow \beta \\ \mathcal{F}_2 \otimes (\mathcal{O}_S \oplus \mathcal{L}) & & \mathcal{F}_2 \\ \downarrow \wr & \xrightarrow{\text{id} \oplus (-\hat{\alpha})} & \downarrow \\ \mathcal{F}_2 \oplus \mathcal{F}_2 \otimes \mathcal{L} & & \mathcal{F}_2 \end{array}$$

The commutativity of this diagram, assuming (2.4), is locally equivalent to (2.8).

$$(2.14) \quad \begin{array}{ccc} \mathcal{F}_1 \otimes \mathcal{F}_1 \otimes \mathcal{F}_1 & \xrightarrow{\text{id} \otimes \beta} & \mathcal{F}_1 \otimes \mathcal{F}_2 \\ \downarrow \beta \otimes \text{id} & & \downarrow \gamma \\ \mathcal{F}_2 \otimes \mathcal{F}_1 & \xrightarrow{\gamma'} & \mathcal{O}_S \oplus \mathcal{L} \end{array}$$

Since $\gamma'(u \otimes v) = \gamma(v \otimes u)$, the commutativity of this diagram is locally equivalent to (2.9) and the analogous one for $(-, -)$, which however follows from (2.4), (2.5), (2.7) and (2.9). Indeed

$$\begin{aligned} (u, \beta(v \otimes w)) &= \langle \alpha(\beta(v \otimes w)), u \rangle = -\langle \beta(v \otimes \alpha(w)), u \rangle = \langle u, \beta(v \otimes \alpha(w)) \rangle \\ &= \langle \alpha(w), \beta(u \otimes v) \rangle = (\beta(u \otimes v), w) = (w, \beta(u \otimes v)) \end{aligned}$$

Remark 2.8. Let A be a commutative (but not necessary associative) ring and $x, y, z \in A$. If

$$(xy)z = x(yz) \text{ and } (yx)z = y(xz)$$

then all the permutations of x, y, z satisfy associativity. Indeed

$$\begin{aligned} y(zx) &= (yx)z = x(yz) = (yz)x, \quad z(xy) = (yx)z = y(xz) = (zx)y \\ (zy)x &= x(yz) = (xy)z = z(yx), \quad (xz)y = y(zx) = (yz)x = x(zy) \end{aligned}$$

Proof. (of Theorem 2.5) We have to show that \mathcal{A} is commutative and associative if and only if $\chi \in \mathcal{Y}$ and it satisfies the properties listed in Theorem 2.5. The “only if” part is an easy consequence of the above discussion. We just highlight some points. Condition (2.5) implies that m is obtained from α as in (2.2), while condition (2.4) implies that $(-, -)$ is obtained from $\langle -, - \rangle$ as in (2.1): in particular $\chi \in \mathcal{Y}$. Finally the symmetry of $(-, -)$ implies that equation (2.6) holds.

We now focus on the converse. So assume $\chi \in \mathcal{Y}$ and that it satisfies the properties listed in Theorem 2.5. By (2.4) and (2.6) we obtain the symmetry of $(-, -)$ and therefore the commutativity of \mathcal{A} .

We need to show that \mathcal{A} is associative. Given $A, B, C \in \{\mathcal{O}_S, \mathcal{L}, \mathcal{F}_1, \mathcal{F}_2\}$ we will say that (A, B, C) holds if $a(bc) = (ab)c$ for all $a \in A, b \in B, c \in C$. Since $\sigma \in \mathbb{Z}/2\mathbb{Z}$ induces a ring automorphism of \mathcal{A} , if (A, B, C) holds then $(\sigma(A), \sigma(B), \sigma(C))$ holds. Moreover, by 2.8, if also (B, A, C) holds then all permutations of (A, B, C) and $(\sigma(A), \sigma(B), \sigma(C))$ hold. Recall that σ fix \mathcal{O}_S and \mathcal{L} and exchanges \mathcal{F}_1 and \mathcal{F}_2 .

Clearly $(\mathcal{L}, \mathcal{L}, \mathcal{L})$ holds. Condition (2.5) insures that $(\mathcal{L}, \mathcal{L}, \mathcal{F}_1)$, $(\mathcal{L}, \mathcal{L}, \mathcal{F}_2)$ and all their permutations hold. Conditions (2.4) and (2.5) say that all the permutations of $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{L})$ hold, while condition (2.7) tells us that all the permutations of $(\mathcal{F}_1, \mathcal{F}_1, \mathcal{L})$ and $(\mathcal{F}_2, \mathcal{F}_2, \mathcal{L})$ hold. The relation (2.8) implies that $(\mathcal{F}_2, \mathcal{F}_2, \mathcal{F}_1)$, $(\mathcal{F}_1, \mathcal{F}_1, \mathcal{F}_2)$ and all their permutations hold. Finally (2.9) says that $(\mathcal{F}_1, \mathcal{F}_1, \mathcal{F}_1)$ and $(\mathcal{F}_2, \mathcal{F}_2, \mathcal{F}_2)$ hold. It is now easy to check that we have obtained all the possible triples. \square

Theorem 2.9. *The functor $\mathcal{Y} \rightarrow \text{LRings}_{\mathcal{R}}^G$ is an equivalence between the substack of \mathcal{Y} of objects making the following diagrams (2.10), (2.12), (2.13), (2.14) and*

$$\begin{array}{ccc} \mathcal{F} \otimes \mathcal{L} \otimes \mathcal{F} & \xrightarrow{\text{id} \otimes \alpha} & \mathcal{F} \otimes \mathcal{F} \\ -\alpha \otimes \text{id} \downarrow & & \downarrow \langle -, - \rangle \\ \mathcal{F} \otimes \mathcal{F} & \xrightarrow{\langle -, - \rangle} & \det \mathcal{F} \end{array}$$

commutative and G -Cov.

Proof. Let $\chi \in \mathcal{Y}$. First of all notice that the commutativity of the diagram in the statement is locally equivalent to (2.6) and, using (2.4), to the symmetry of $(-, -)_{\chi}$. By definition of \mathcal{Y} and 2.2.1, this is also equivalent to the commutativity of \mathcal{A}_{χ} . Thus the claim follows from 2.5 and 2.2.2. \square

2.3. Local analysis. Let $\chi = (\mathcal{L}, \mathcal{F}, \alpha, \beta, \langle -, - \rangle) \in \mathcal{Y}$ and assume that $t \in \mathcal{L}$ is a generator and that y, z is a basis of \mathcal{F} . The aim of this subsection is to translate conditions (2.5), (2.6), (2.7), (2.8) and (2.9), writing all the maps $\alpha, \beta, \langle -, - \rangle$ with respect to the given basis. In particular we will use notation from 2.4, so that $m \in \mathcal{O}_T$, α is a map $\mathcal{F} \rightarrow \mathcal{F}$ and $\langle -, - \rangle: \det \mathcal{F} \rightarrow \mathcal{O}_T$.

Notation 2.10. Write

$$\beta(y^2) = ay + bz, \quad \beta(yz) = cy + dz, \quad \beta(z^2) = ey + fz, \quad \langle y, z \rangle = \omega, \quad \alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

In particular:

$$(y, y) = -C\omega, \quad (y, z) = -D\omega, \quad (z, y) = A\omega, \quad (z, z) = B\omega, \quad m = (A^2 + D^2)/2 + BC$$

Lemma 2.11. *The object $\chi = (\mathcal{L}, \mathcal{F}, \alpha, \beta, \langle -, - \rangle) \in \mathcal{Y}$ belongs to G -Cov if and only if the following relations hold.*

$$(2.15) \quad \begin{array}{ll} (2.5) & \iff (A - D)(A + D) = B(A + D) = C(A + D) = 0 \\ (2.6) & \iff \omega(A + D) = 0 \\ (2.7) & \iff \begin{cases} (2aA + bB + cC) = (2cA + dB + eC) = 0 \\ C(a + d) + b(A + D) = C(c + f) + d(A + D) = 0 \\ B(a + d) + c(A + D) = B(c + f) + e(A + D) = 0 \\ a(A + D) - D(a + d) = c(A + D) - D(c + f) = 0 \end{cases} \\ (2.8) & \iff \begin{cases} a^2 + bc = -\omega C, \quad ac + be = \omega(A - D), \quad c^2 + de = B\omega \\ (a - d)(a + d) = b(a + d) = c(a + d) = 0 \\ (c - f)(c + f) = d(c + f) = e(c + f) = 0 \\ a(a + d) + b(c + f) = e(a + d) + c(c + f) = 0 \end{cases} \\ (2.9) & \iff \omega(a + d) = \omega(c + f) = 0 \end{array}$$

Proof. The claims follow from the following relations, which can be computed directly.

$$\begin{aligned} \langle z, \beta(y^2) \rangle &= -a\omega & \langle y, \beta(zy) \rangle &= \langle y, \beta(yz) \rangle = d\omega \\ \langle y, \beta(z^2) \rangle &= f\omega & \langle z, \beta(yz) \rangle &= \langle z, \beta(zy) \rangle = -c\omega \end{aligned}$$

$$\begin{aligned} \alpha(\beta(y^2)) + \beta(y\alpha(y)) &= (2aA + bB + cC)y + (C(a + d) + b(A + D))z \\ \alpha(\beta(zy)) + \beta(z\alpha(y)) &= (2cA + dB + eC)y + (C(c + f) + d(A + D))z \\ \alpha(\beta(yz)) + \beta(y\alpha(z)) &= (B(a + d) + c(A + D))y + (2dD + bB + cC)z \\ \alpha(\beta(z^2)) + \beta(z\alpha(z)) &= (B(c + f) + e(A + D))y + (2fD + eC + dB)z \end{aligned}$$

If we set $\Gamma(u, v, w) = \langle \alpha(w), v \rangle u + \langle v, w \rangle \alpha(u) = (v, w)u + \langle v, w \rangle \alpha(u)$ we have

$$\begin{aligned} \beta(\beta(y^2)y) &= (a^2 + bc)y + b(a + d)z & \Gamma(y, y, y) &= -C\omega y \\ \beta(\beta(y^2)z) &= (ac + be)y + (ad + bf)z & \Gamma(y, y, z) &= \omega(A - D)y + \omega Cz \\ \beta(\beta(z^2)y) &= (ea + fc)y + (eb + fd)z & \Gamma(z, z, y) &= -B\omega y + \omega(A - D)z \\ \beta(\beta(z^2)z) &= e(c + f)y + (ed + f^2)z & \Gamma(z, z, z) &= B\omega z \\ \beta(\beta(yz)y) &= \beta(\beta(zy)y) = c(a + d)y + (cb + d^2)z & \Gamma(y, z, y) &= \Gamma(z, y, y) = -C\omega z \\ \beta(\beta(yz)z) &= \beta(\beta(zy)z) = (c^2 + de)y + d(c + f)z & \Gamma(y, z, z) &= \Gamma(z, y, z) = B\omega y \end{aligned}$$

□

Notation 2.12. Given $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -, - \rangle) \in \mathcal{Y}$, a basis y, z of \mathcal{F} and a generator $t \in \mathcal{L}$, we will denote by

$$a_\chi, b_\chi, c_\chi, d_\chi, e_\chi, f_\chi, \omega_\chi, A_\chi, B_\chi, C_\chi, D_\chi, m_\chi$$

the data associated with χ as above. We will always omit the $-\chi$ if this will not lead to confusion.

2.4. From $(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$ -covers to S_3 -covers. In this section we compare G -covers and S_3 -covers. We denote by \mathcal{R}_3 the ring of integers with 6 inverted, that is $\mathcal{R}_3 = \mathbb{Z}[1/6] = \mathcal{R}[1/3]$. Theorem 1.5 for $n = 3$ applies to G and $H = \mathbb{Z}/3\mathbb{Z} \rtimes (\mathbb{Z}/3\mathbb{Z})^*$ over $\mathcal{R}_3 \supseteq \mathbb{Z}[1/3]$. We fix the isomorphism $H \rightarrow S_3$ mapping $(1, 1) \mapsto (123)$, $(0, 2) \mapsto (12)$.

Proposition 2.13. *Over \mathcal{R}_3 the equivalence $G\text{-Cov} \rightarrow S_3\text{-Cov}$ of 1.5 maps a G -cover $\mathcal{A} = \mathcal{O} \oplus \mathcal{L} \oplus (V^\vee \otimes \mathcal{F})$ (where $\mathcal{F}_1 = v_2^* \otimes \mathcal{F}$ and $\mathcal{F}_2 = v_1^* \otimes \mathcal{F}$) to the sub algebra*

$$\mathcal{C} = \mathcal{O} \oplus \mathcal{L}z \oplus [(v_2^*x + v_1^*x^2) \otimes \mathcal{F}] \oplus [(v_2^*zx - v_1^*zx^2) \otimes \mathcal{F}] \subseteq \mathcal{A}[x, z]/(x^3 - 1, z^2 + 3)$$

The left S_3 -action on the above algebras is trivial on \mathcal{A} and satisfies

$$(123)x = x(z - 1)/2, (123)z = z, (12)x = x, (12)z = -z$$

Over $Y = \text{Spec}(\mathcal{R}_3[w]/(w^2 + 3))$ the map $\mathcal{C} \rightarrow \mathcal{A}, x \mapsto 1, z \mapsto w$ is an isomorphism equivariant with respect to the isomorphism $(S_3)_Y \rightarrow G_Y$, $(123) \mapsto (w - 1)/2$, $(12) \mapsto \sigma$.

Proof. We are going to rewrite 1.5 in this simplified situation. We have $\Phi_3(y) = 1 + y + y^2$ and, over \mathcal{R}_3 ,

$$K = \mathcal{R}_3[y]/(y^2 + y + 1) = \mathcal{R}_3[z]/(z^2 + 3) = \mathcal{R}_3 \oplus \mathcal{R}_3z \text{ where } z = 1 + 2y, y = (z - 1)/2$$

Moreover $\mu_3^* = \text{Spec } K$ and the natural involution of μ_3^* acts as $y \mapsto y^2$ or $z \mapsto -z$. In particular $\mathcal{A} \otimes A_P = \mathcal{A}[x, z]/(x^3 - 1, z^2 + 3)$ and its S_3 -action is

$$(123) \leftrightarrow (1, 1): x \mapsto xy = x(z - 1)/2, z \mapsto z, (12) \leftrightarrow (0, 2): x \mapsto x, z \mapsto -z$$

Instead σ acts on $\mathcal{A} \otimes A_P$ with the usual action on \mathcal{A} , while on x, z , since $G \ni \sigma \mapsto (1, 2) \in \mu_3 \rtimes (\mathbb{Z}/3\mathbb{Z})^*$, as $x \mapsto x^2, z \mapsto -z$. The μ_3 -action on $\mathcal{A} \otimes A_P$ is the usual one on \mathcal{A} , while on A_P we have $\deg x = -1 = 2$ and $\deg z = 0$. In other words $A_P = K \oplus Kx^2 \oplus Kx$ is the graduation induced by μ_3 . We have

$$(\mathcal{A} \otimes A_P)^{\mu_3} = [(\mathcal{O} \oplus \mathcal{L}) \otimes K] \oplus [v_2^* \otimes \mathcal{F} \otimes Kx] \oplus [v_1^* \otimes \mathcal{F} \otimes Kx^2]$$

In particular $(\mathcal{A} \otimes A_P)^G = \mathcal{O} \oplus \mathcal{L}z \oplus W$ where W is the σ -invariant of $[v_2^* \otimes \mathcal{F} \otimes Kx] \oplus [v_1^* \otimes \mathcal{F} \otimes Kx^2]$. As σ exchanges the two factors we have that W is the sheaf of elements of the form

$$v_2^* \otimes f \otimes \lambda x + \sigma(v_2^* \otimes f \otimes \lambda x) = v_2^* \otimes f \otimes \lambda x + v_1^* \otimes f \otimes \sigma(\lambda)x^2 \text{ for } f \in \mathcal{F}, \lambda \in K$$

Since $K = \mathcal{R}_3 \oplus \mathcal{R}_3z$, for $\lambda = 1$ and $\lambda = z$ we obtain

$$v_2^* \otimes f \otimes x + v_1^* \otimes f \otimes x^2 = (v_2^*x + v_1^*x^2) \otimes f \text{ and } v_2^* \otimes f \otimes zx + v_1^* \otimes f \otimes -zx^2 = (v_2^*zx - v_1^*zx^2) \otimes f$$

The last statement follows directly from 1.5. □

Remark 2.14. It is a classical result that $\text{Et}_3 \rightarrow \text{B } S_3$, $Q \mapsto \underline{\text{Iso}}(\mathbb{Z}/3\mathbb{Z}, Q)$, where Et_3 is the stack of degree 3 étale covers and $\mathbb{Z}/3\mathbb{Z}$ is thought of as a scheme, is an equivalence (see [?, Prop 2.7]). Thus over \mathcal{R}_3 the S_3 -torsor $P = \mu_3 \times \mu_3^*$ of 1.5 is induced by a degree 3 étale cover over \mathcal{R}_3 : this is $\mu_3 \rightarrow \text{Spec } \mathcal{R}_3$ itself. Indeed a direct check shows that

$$P = \mu_3 \times \underline{\text{Iso}}_{\text{groups}}(\mathbb{Z}/3\mathbb{Z}, \mu_3) \rightarrow \underline{\text{Iso}}_{\text{sets}}(\mathbb{Z}/3\mathbb{Z}, \mu_3), (g, \phi) \mapsto m_g \circ \phi$$

where m_g denotes the multiplication by an element of $g \in \mu_3$, is an S_3 -equivariant map of S_3 -torsors, hence an isomorphism.

3. GEOMETRY OF $(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$ -COV AND S_3 -COV

We keep the notation from Section 2. In particular $\mathcal{R} = \mathbb{Z}[1/2]$, $\mathcal{R}_3 = \mathbb{Z}[1/6]$ and $G = \mu_3 \rtimes \mathbb{Z}/2\mathbb{Z}$. The aim of this section is to describe the geometry of the stacks G -Cov over \mathcal{R} . In particular, over \mathcal{R}_3 , we obtain a description of S_3 -Cov $\simeq G$ -Cov thanks to 2.13.

3.1. A smooth atlas for $(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$ -Cov. Set

$$P = \mathcal{R}[a, b, c, d, e, f, A, B, C, D, \omega] / (\text{relations 2.11})$$

By 2.11 the scheme $\text{Spec } P$ represents the functor $(\text{Sch}/\mathcal{R})^{\text{op}} \rightarrow (\text{Sets})$ which with any \mathcal{R} -scheme associates the setoid of G -covers $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -, - \rangle)$ with a given basis for \mathcal{L} and \mathcal{F} . In other words $\text{Spec } P$ is isomorphic to the fiber product (over \mathcal{R}) of $(\mathcal{L}, \mathcal{F}): G\text{-Cov} \rightarrow (\text{B } \mathbb{G}_m \times \text{B } \text{GL}_2)$ and the trivial torsor $\text{Spec } \mathcal{R} \rightarrow \text{B } \mathbb{G}_m \times \text{B } \text{GL}_2$. The map $\text{Spec } P \rightarrow G\text{-Cov}$ corresponds to

$$(P, P^2, \alpha, \beta, \langle -, - \rangle) \text{ where } \alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \beta = \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix}, \langle -, - \rangle = \omega$$

Here we are using the canonical basis of P and P^2 . In particular

Theorem 3.1. *The map $\text{Spec } P \rightarrow G\text{-Cov}$ is a $(\mathbb{G}_m \times \text{GL}_2)$ -torsor and*

$$G\text{-Cov} \simeq [\text{Spec } P / (\mathbb{G}_m \times \text{GL}_2)]$$

We now want to discuss the ring P more in details and, in particular, prove the following result:

Theorem 3.2. *The ring P is a flat \mathcal{R} -algebra, it has two minimal primes*

$$Q_1 = (a + c, d + f, A + D) \text{ and } Q_2 = (a, b, c, d, e, f, B, C, A - D, \omega)$$

$\text{Spec}(P/Q_1) \rightarrow \text{Spec } \mathcal{R}$ is flat and geometrically integral and $\text{Spec}(P/Q_2) \simeq \mathbb{A}_{\mathcal{R}}^1$.

Moreover $Q_1 + Q_2 = (a, b, c, d, e, f, A, B, C, D, \omega)$, $Q_1 \cap Q_2 = (a + c, d + f) = \sqrt{0_P}$ and $\text{Spec}(P)^{\text{red}} \rightarrow \text{Spec } \mathcal{R}$ is flat and geometrically reduced.

Definition 3.3. We denote by $\{0\}$ the closed substack of $\chi \in G\text{-Cov}$ such that $\alpha_\chi = \beta_\chi = \langle -, - \rangle_\chi = 0$.

Remark 3.4. The inclusion $\{0\} \rightarrow G\text{-Cov}$, over the atlas $\text{Spec } P \rightarrow G\text{-Cov}$, corresponds to the map $\text{Spec } \mathcal{R} \rightarrow \text{Spec } P$ mapping all variables to 0. In particular it follows that $\{0\} \simeq \text{B } \mathbb{G}_m \times \text{B } \text{GL}_2$.

A direct consequence of 3.1 and 3.2 is the following.

Theorem 3.5. *The stack $G\text{-Cov}$ is a flat and finitely generated algebraic stack over \mathcal{R} , the closed substacks (see B.4)*

$$\mathcal{Z}_G = \{\chi \mid \text{tr } \beta_\chi = \text{tr } \alpha_\chi = 0\}, \mathcal{Z}_2 = \{\chi \mid \beta_\chi = \langle -, - \rangle_\chi = 0, \alpha_\chi = ((\text{tr } \alpha_\chi)/2) \otimes \text{id}_{\mathcal{F}_\chi}\}$$

are the irreducible components of $G\text{-Cov}$ with their reduced structure and they are flat and geometrically integral over \mathcal{R} . Moreover $\mathcal{Z}_G \cap \mathcal{Z}_2 = \{0\}$, $(G\text{-Cov})^{\text{red}} = \{\chi \mid \text{tr } \beta_\chi = 0\}$ and it is flat and geometrically reduced over \mathcal{R} . Finally we have a decomposition into open substacks

$$(3.1) \quad (G\text{-Cov} - \{0\}) = (\mathcal{Z}_G - \{0\}) \sqcup (\mathcal{Z}_2 - \{0\})$$

We collect some partial results before proving Theorem 3.2.

Lemma 3.6. *We have $(a+d)^3 = (c+f)^3 = 0$ in P . On the other hand $a+d$ and $c+f$ are not zero in any geometric fiber of $\text{Spec } P \rightarrow \text{Spec } \mathcal{R}$.*

Proof. Indeed, by (2.15), we have $a(a+d) = d(a+d)$ and

$$0 = [a(a+d) + b(c+f)](a+d) = a(a+d)^2 = d(a+d)^2 \implies (a+d)^3 = 0$$

The expression $(c+f)^3 = 0$ is proved similarly.

For the second claim, if k is an algebraically closed field over \mathcal{R} , the choice

$$a = b = c = d = e = f = \omega = A = B = C = D = x$$

defines a map $P \rightarrow k[x]/(x^2)$ sending $a+d$ and $c+f$ to x . □

Lemma 3.7. *The map $P \rightarrow P_{A+D}$ has kernel Q_2 of Theorem 3.2 and factors as*

$$P \twoheadrightarrow \mathcal{R}[A] \hookrightarrow \mathcal{R}[A]_A, \quad a, b, c, d, e, f, \omega, B, C \mapsto 0, D \mapsto A, A \mapsto A$$

Proof. Clearly $P/Q_2 \simeq \mathcal{R}[A]$, so it is enough to show that $a, b, c, d, e, f, \omega, B, C, A - D$ are zero in P_{A+D} . We make use of (2.15). We immediately have $B = C = \omega = 0$ and $A = D$, which is therefore invertible in P_{A+D} . From the local equations corresponding to (2.7) we first obtain $b = c = d = e = 0$, then $2aA = 0$ and therefore, as A is invertible, $a = 0$. Finally the last equation yields $Df = Af = 0$ and thus $f = 0$. □

We now focus on the first prime ideal Q_1 .

Lemma 3.8. *Let D be a ring and a, b_1, \dots, b_m be a regular sequence in D . Then a is a non zero divisor in $D[X_1, \dots, X_m]/(aX_i - b_i \mid i \leq m)$.*

Proof. Denote by S_m the last ring in the statement. We proceed by induction on m .

Base case $m = 1$. So let $f \in D[X_1]$ such that $af = 0$ in S_1 , so that

$$af(X_1) = g(X_1)(aX_1 - b_1) \text{ in } D[X_1] \text{ for some } g \in D[X_1]$$

It follows that $b_1g(X_1) = 0$ in $(D/(a))[X_1]$ and, by hypothesis, that $a \mid g(X_1)$ in $D[X_1]$, say $g = a\tilde{g}$. As a is a non zero divisor in $D[X_1]$ we can conclude that $f = \tilde{g}(aX_1 - b_1)$, that is $f = 0$ in S_1 .

Inductive step $m \implies m + 1$. From the base case it is enough to show that a, b_{m+1} is a regular sequence in S_m . From the case m we know that a is a non zero divisor in S_m . On the other hand

$$S_m/(a) = D[X_1, \dots, X_m]/(a, b_1, \dots, b_m) = (D/(a, b_1, \dots, b_m))[X_1, \dots, X_m]$$

which shows that b_{m+1} is a non zero divisor on $S_m/(a)$. □

Remark 3.9. Let D be a ring and $b_1, \dots, b_m \in D$. Then the kernel of the map

$$D[X_1, \dots, X_m] \rightarrow D[t], \quad X_i \mapsto tb_i$$

is the graded ideal generated by the homogeneous polynomials f such that $f(b_1, \dots, b_m) = 0$.

Lemma 3.10. *Let D be a ring, $b_1, \dots, b_m \in D$ and denote by J the graded ideal of $D[X_1, \dots, X_m]$ generated by the homogeneous polynomials f such that $f(b_1, \dots, b_m) = 0$. Consider also the ring*

$$S = \frac{D[X_1, \dots, X_m, \omega]}{(J, \omega X_i - b_i)}$$

Then $S_\omega = D[\omega]_\omega$ and the map

$$S \rightarrow D[\omega]_\omega, X_i \mapsto b_i/\omega$$

is well defined and injective.

Proof. The ring S_ω is the quotient of $D[\omega]_\omega$ by the ideal generated by $f(\underline{b}/\omega)$ for $f \in J$, where $\underline{b} = (b_1, \dots, b_m)$. Since $f(\underline{b}/\omega) = f(\underline{b})/\omega^d = 0$ if $f \in D[X_1, \dots, X_m]$ is an homogeneous polynomial of degree d and $f \in J$, we can conclude that $S_\omega = D[\omega]_\omega$. Consider the map

$$\phi: D[X_1, \dots, X_m, \omega]/J \rightarrow D[\omega, t], X_i \mapsto tb_i$$

It is injective by 3.9. We have to prove that the kernel \mathcal{K} of the map

$$D[X_1, \dots, X_m, \omega]/J \rightarrow D[\omega, t] \rightarrow D[\omega, t]/(\omega t - 1) \simeq D[\omega]_\omega, X_i \mapsto b_i/\omega$$

is generated by the $\omega X_i - b_i$. Thus let $f \in \mathcal{K}$, which just means $\phi(f) \in (\omega t - 1)$. Since $\phi(\omega X_i - b_i) = 0$ we can assume

$$f = \sum_{\alpha \in \mathbb{N}^m} f_\alpha \underline{X}^\alpha \text{ with } f_0 \in D[\omega] \text{ and } f_\alpha \in D \text{ if } |\alpha| > 0$$

By assumption

$$\phi(f) = \sum_{\alpha \in \mathbb{N}^m} f_\alpha t^{|\alpha|} \underline{b}^\alpha = g(\omega, t)(\omega t - 1) \implies \sum_{\alpha \mid |\alpha|=l} f_\alpha \underline{b}^\alpha = g_{l-1}\omega - g_l \text{ for } l \geq 0$$

where $g = \sum_{l \in \mathbb{Z}} g_l t^l \in D[\omega][t]$, that is with $g_l = 0$ if $l < 0$ or $l \gg 0$. We claim that $g = 0$ and, therefore, $f = 0$. Indeed, otherwise, if q is the degree of g with respect to t , we would have

$$g_q \omega = \sum_{\alpha \mid |\alpha|=q+1} f_\alpha \underline{b}^\alpha \in D \implies g_q = 0$$

□

Proposition 3.11. *Let $S = \mathcal{R}[a, b, c, e, A, B, C, \omega]/I$ where I is the ideal*

$$I = (2aA + bB + cC, 2cA - aB + eC, 2\omega A - (ac + be), \omega B - (c^2 - ae), \omega C + a^2 + bc)$$

Then $\text{Spec } S \rightarrow \text{Spec } \mathcal{R}$ is flat and geometrically integral.

Proof. Flatness follows from geometric integrality. Indeed this is equivalent to the torsion freeness of S , which, in case S is a domain, coincides with the injectivity of $\mathcal{R} \rightarrow S$. This follows from the existence of the morphism $S \rightarrow \mathcal{R}$ mapping all variables to 0.

We now have to prove that, if we replace \mathcal{R} by any domain R , the ring S is a domain. Consider $b_1 = (ac + be)/2$, $b_2 = (c^2 - ae)$ and $b_3 = -a^2 - bc$ and $D = R[a, b, c, e]$. By 3.9 and 3.10 we have to prove that, if

$$J = (2aX_1 + bX_2 + cX_3, 2cX_1 - aX_2 + eX_3)$$

then $\phi: P = D[X_1, X_2, X_3]/J \rightarrow D[t], X_i \mapsto tb_i$ is well defined and injective. It is well defined because

$$\det \begin{pmatrix} Y_1 & Y_2 & Y_3 \\ 2a & b & c \\ 2c & -a & e \end{pmatrix} = 2(Y_1 b_1 + Y_2 b_2 + Y_3 b_3)$$

Thus we focus on the injectivity. Over D_e we have

$$P_e = D_e[X_1, X_2, X_3]/J = D_e[X_1, X_2]/(b_1X_2 - b_2X_1)$$

It follows that ϕ is an isomorphism over D_{eb_1} . Since $eb_1 \neq 0$, the claim holds if we show that P is a domain. We apply 3.8 twice. We have that

$$P_1 = D[a, b, c, X_1, X_2][X_3]/(cX_3 - (-2aX_1 - bX_2))$$

is a subdomain of $(P_1)_c = D[a, b, c, X_1, X_2]_c$ because $c, 2aX_1 + bX_2$ is a regular sequence in $D[a, b, c, X_1, X_2]$. Now consider

$$P = P_1[e]/(X_3e - (aX_2 - 2cX_1))$$

We prove that $f = aX_2 - 2cX_1$ is a non zero divisor in $P_1/(X_3)$, so that P would be a subdomain of $P_{X_3} = (P_1)_{X_3} = D[a, b, X_1, X_2, X_3]_{X_3}$. Set $g = 2aX_1 + bX_2$. The map

$$P_1/(X_3) = (D[a, b, X_1, X_2]/(g))[c] \rightarrow (P_1)_c/(X_3) = (P_1)_c/(g)$$

is injective, thus we have to exclude that f is a zero divisor in the bigger ring. Consider the change of variable

$$\bar{X}_1 = -f/(2c) = X_1 - (a/(2c))X_2 \implies g = 2a\bar{X}_1 - b_3X_2, (P_1)_c = D[a, b, c, \bar{X}_1, X_2]_c$$

If f , and so \bar{X}_1 , is a zero divisor in $(P_1)_c/(g)$ we have

$$\bar{X}_1 p = gq \text{ in } (P_1)_c \implies gq = b_3X_2q = 0 \text{ in } (P_1)_c/(\bar{X}_1) = D[a, b, c, X_2]_c \implies q = \tilde{q}\bar{X}_1$$

and thus $p = \tilde{q}g$, that is $p = 0$ in $(P_1)_c/(g)$. \square

Proof of Theorem 3.2. From 3.7 we know that Q_2 is a prime ideal, $P/Q_2 \simeq \mathcal{R}[A]$, $\text{Spec}(P/Q_2)$ is the schematic closure of the open immersion $\text{Spec } P_{A+D} \rightarrow \text{Spec } P$ and therefore that Q_2 is a minimal prime. From 3.11, since $S/I = P/Q_1$, we deduce that Q_1 is a prime and $\text{Spec}(P/Q_1) \rightarrow \text{Spec } \mathcal{R}$ is flat and geometrically integral. Since Q_1 goes to zero along the map $P \rightarrow P_\omega$, it has to be the kernel and therefore a minimal prime.

From 3.6 we have $(a+d, c+f) \subseteq \sqrt{0_P} \subseteq Q_1 \cap Q_2$. In order to check the equality, we can work over $P' = P/(a+d, c+f)$. Denote by Q'_1, Q'_2 the images of Q_1, Q_2 respectively. In particular $Q'_1 = (A+D)$ and $Q'_1Q'_2 = 0$. If $u \in Q'_1 \cap Q'_2$ in P' , we have $u = (A+D)v \in Q'_2$. Since $A+D \notin Q'_2$ it follows that $v \in Q'_2$. Hence $u \in Q'_1Q'_2 = 0$. In conclusion $Q_1 \cap Q_2 = (a+d, c+f) = \sqrt{0_P}$ and Q_1, Q_2 are the only minimal primes of P . Moreover, since $Q_1 + Q_2$ is the kernel of the “zero map” $P \rightarrow \mathcal{R}$, there is an exact sequence of \mathcal{R} -modules

$$0 \rightarrow P/(Q_1 \cap Q_2) \rightarrow P/Q_1 \times P/Q_2 \rightarrow (P/Q_1 + Q_2) \simeq \mathcal{R} \rightarrow 0$$

It follows that $Q_1 \cap Q_2$ commutes with arbitrary base changes of \mathcal{R} and, since $\text{Spec } P/Q_1$ and $\text{Spec } P/Q_2$ are geometrically integral over \mathcal{R} , we can conclude that $\text{Spec}(P/(Q_1 \cap Q_2)) = (\text{Spec } P)^{\text{red}} \rightarrow \text{Spec } \mathcal{R}$ is geometrically reduced. From the above sequence, since P/Q_1 and P/Q_2 are \mathcal{R} -flat, we can also conclude that $P/(Q_1 \cap Q_2)$ is \mathcal{R} -flat. This also implies that, if $x \in P$ is a non-zero divisor as \mathcal{R} -module, then $x \in Q_1 \cap Q_2 = \sqrt{0_P}$, that is x is nilpotent in P . Applying the “zero map” $P \rightarrow \mathcal{R}$ it follows that x is nilpotent in \mathcal{R} and thus $x = 0$. In conclusion P is \mathcal{R} -torsion free and therefore \mathcal{R} -flat. \square

Lemma 3.12. *The action $\text{Spec } P \times \mathbb{G}_m \rightarrow \text{Spec } P$ obtained restricting the $\text{GL}_2 \times \mathbb{G}_m$ -action on P via $j: \mathbb{G}_m \rightarrow \text{GL}_2 \times \mathbb{G}_m$, $j(\lambda) = (\text{lid}, \lambda)$ is given by $(\chi, \lambda) \mapsto \lambda^{-1}\chi$.*

Proof. We think of $F = \text{Spec } P$ as the functor $(\text{Sch}/\mathcal{R})^{\text{op}} \rightarrow (\text{Sets})$ such that $F(T)$ is the set of sequences $\chi = (\mathcal{O}_T, \mathcal{O}_T^2, \alpha, \beta, \langle -, - \rangle)$ belonging to $G\text{-Cov}(T)$. If $\chi = (\mathcal{O}_T, \mathcal{O}_T^2, \alpha, \beta, \langle -, - \rangle) \in F(T)$ and $(M, \lambda) \in \text{GL}_2 \times \mathbb{G}_m(T)$ then $\chi \cdot (M, \lambda) = (\mathcal{O}_T, \mathcal{O}_T^2, \alpha', \beta', \langle -, - \rangle') \in F(T)$ is the unique element such that (M, λ) defines a morphism $\chi \rightarrow \chi \cdot (M, \lambda)$ in $G\text{-Cov}(T)$. In other words we must have: $M\alpha = \alpha'(\lambda \otimes M)$, $M\beta = \beta' \text{Sym}^2 M$ and $\lambda \langle -, - \rangle = \langle -, - \rangle' \det M$. If $M = \text{lid}$ a

simple computation shows that $\alpha' = \lambda^{-1}\alpha$, $\beta' = \lambda^{-1}\beta$ and $\langle -, - \rangle' = \lambda^{-1}\langle -, - \rangle$, which implies the claim. \square

Theorem 3.13. *The GL_2 -torsor induced by $\mathcal{F} \otimes \mathcal{L}^{-1}: (G\text{-Cov} - \{0\}) \rightarrow \mathrm{BGL}_2$ is of the form $\mathrm{Proj} P \rightarrow (G\text{-Cov} - \{0\})$, where P has the natural grading of a polynomial algebra, and it splits accordingly to (3.1) as $\mathrm{Proj} P = \mathrm{Proj} P_1 \sqcup \mathrm{Spec} \mathcal{R}$, where P_1 is the algebra of 3.11. In particular we have presentations*

$$(G\text{-Cov} - \{0\}) \simeq [\mathrm{Proj} P / \mathrm{GL}_2], \quad (\mathcal{Z}_G - \{0\}) \simeq [\mathrm{Proj}(P_1) / \mathrm{GL}_2], \quad (\mathcal{Z}_2 - \{0\}) \simeq \mathrm{BGL}_2$$

Proof. Consider the Cartesian diagrams

$$\begin{array}{ccccc} \mathrm{Spec} P - \{0\} & \longrightarrow & \mathrm{Spec} \mathcal{R} & & \\ \downarrow & & \downarrow & & \\ V & \longrightarrow & \mathrm{B}\mathbb{G}_m & \longrightarrow & \mathrm{Spec} \mathcal{R} \\ \downarrow & & \downarrow \mathrm{B}(j) & & \downarrow \\ G\text{-Cov} - \{0\} & \xrightarrow{(\mathcal{F}, \mathcal{L})} & \mathrm{B}(\mathrm{GL}_2 \times \mathbb{G}_m) & \xrightarrow{\mathrm{B}(\phi)} & \mathrm{B}(\mathrm{GL}_2) \end{array}$$

Here $\{0\} \subseteq \mathrm{Spec} P$ is the closed subscheme defined by the graded irrelevant ideal. The map $\phi: \mathrm{GL}_2 \times \mathbb{G}_m \rightarrow \mathrm{GL}_2$ is $\phi(M, \lambda) = \lambda^{-1}M$, which is a surjective group homomorphism whose kernel is $j: \mathbb{G}_m \rightarrow \mathrm{GL}_2 \times \mathbb{G}_m$, $j(\lambda) = (\lambda \mathrm{id}, \lambda)$. The bottom right diagram is Cartesian thanks to [HMT20, Cor 1.21]. The horizontal arrow $G\text{-Cov} - \{0\} \rightarrow \mathrm{BGL}_2$ at the bottom is exactly induced by $\mathcal{F} \otimes \mathcal{L}^{-1}$. It follows that $V \simeq [\mathrm{Spec} P - \{0\} / \mathbb{G}_m]$ where the action of \mathbb{G}_m is the restriction along j of the action of $\mathrm{GL}_2 \times \mathbb{G}_m$ on $\mathrm{Spec} P$. By 3.12 we can conclude that $[\mathrm{Spec} P - \{0\} / \mathbb{G}_m] \simeq \mathrm{Proj} P$. For the last part notice that $P/Q_1 = P_1$ and $P/Q_2 = \mathcal{R}[A]$, so that $\mathrm{Proj}(P/Q_2) = \mathrm{Spec} \mathcal{R}$ (see 3.2). As Q_1 and Q_2 are graded ideals also the last remaining claims holds. \square

Corollary 3.14. *The stacks $(G\text{-Cov} - \{0\}) \rightarrow \mathrm{Spec} \mathcal{R}$ and $(\mathcal{Z}_G - \{0\}) \rightarrow \mathrm{Spec} \mathcal{R}$ are universally closed morphisms of stacks.*

Proof. Both stacks $f: \mathcal{X} \rightarrow \mathrm{Spec} \mathcal{R}$ admits a smooth atlas $U \rightarrow \mathcal{X}$ such that $U \rightarrow \mathrm{Spec} \mathcal{R}$ is universally closed thanks to 3.13. It follows easily that $f: \mathcal{X} \rightarrow \mathrm{Spec} \mathcal{R}$ is universally closed as well. \square

3.2. Exceptional irreducible component of $(\mu_3 \times \mathbb{Z}/2\mathbb{Z})$ -Cov. In this section we describe the irreducible component $\mathcal{Z}_2 = \{\chi \mid \beta_\chi = \langle -, - \rangle_\chi = 0, \alpha_\chi = ((\mathrm{tr} \alpha_\chi)/2) \otimes \mathrm{id}_{\mathcal{F}_\chi}\}$ of $G\text{-Cov}$ (see 3.5).

Theorem 3.15. *Let \mathcal{Z}'_2 be the stack over \mathcal{R} of triples $(\mathcal{L}, \mathcal{F}, \mu)$ where \mathcal{L} is an invertible sheaf, \mathcal{F} is a rank 2 locally free sheaf and $\mu: \mathcal{L} \rightarrow \mathcal{O}$ is a map. Then $\mathcal{Z}'_2 \simeq [\mathbb{A}^1_{\mathcal{R}} / \mathbb{G}_m] \times \mathrm{BGL}_2$ and we have an equivalence*

$$\mathcal{Z}'_2 \rightarrow \mathcal{Z}_2, \quad (\mathcal{L}, \mathcal{F}, \mu) \mapsto (\mathcal{L}, \mathcal{F}, \mu \otimes \mathrm{id}_{\mathcal{F}}, 0, 0)$$

Moreover $\mathcal{Z}_2 - \{0\}$ is an open substack of $G\text{-Cov}$ and it is equivalent to BGL_2 .

Proof. The last claim is included in 3.13. Since BGL_2 is the stack of rank 2 locally free sheaves and $[\mathbb{A}^1 / \mathbb{G}_m]$ is the stack of invertible sheaves with a section, it follows that $\mathcal{Z}'_2 \simeq [\mathbb{A}^1_{\mathcal{R}} / \mathbb{G}_m] \times \mathrm{BGL}_2$. It is easy to check locally that the functor $\mathcal{Z}'_2 \rightarrow \mathcal{Z}_2$ is well defined and, using B.5, that it is an equivalence. \square

3.3. Smooth locus of $(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})$ -Cov.

Theorem 3.16. *We have*

$$G\text{-Cov} - \{0\} \simeq \text{B GL}_2 \sqcup (\mathcal{Z}_G - \{0\})$$

and it is the smooth locus of $G\text{-Cov} \rightarrow \text{Spec } \mathcal{R}$.

We state here the above theorem to be consistent with the section argument. Anyway we will prove it in Section 4.5, after showing the smoothness of $\mathcal{Z}_G - \{0\}$ over \mathcal{R} . Here we show that $\{0\}$ cannot be smooth:

Lemma 3.17. *The closed substack $\{0\}$ does not meet the smooth locus of any of \mathcal{Z}_G , $G\text{-Cov}$ or $(G\text{-Cov})^{\text{red}}$ (see 3.5).*

Proof. By 3.2 any of these stacks \mathcal{X} has a smooth atlas $f: \text{Spec } S \rightarrow \mathcal{X}$ of the form $S = \mathcal{R}[y_1, \dots, y_m]/(q_1, \dots, q_l)$, with q_i quadratic polynomials, and such that $f^{-1}\{0\} = V(y_1, \dots, y_m)$, the zero locus of the y_i . Moreover $l \geq 1$. It follows that the Jacobian ideal in S is contained in (y_1, \dots, y_m) , which proves the claim. \square

4. GEOMETRY OF THE MAIN IRREDUCIBLE COMPONENTS $\mathcal{Z}_{(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z})}$ AND \mathcal{Z}_{S_3}

We keep the notation from Section 2. In particular $\mathcal{R} = \mathbb{Z}[1/2]$, $\mathcal{R}_3 = \mathbb{Z}[1/6]$ and $G = \mu_3 \rtimes \mathbb{Z}/2\mathbb{Z}$. The aim of this section is to describe the geometry of the irreducible component \mathcal{Z}_G of $G\text{-Cov}$ over \mathcal{R} . In particular, over \mathcal{R}_3 , we obtain a description of $\mathcal{Z}_{S_3} \simeq \mathcal{Z}_G$ thanks to 2.13. We proceed by looking at particular open substacks of \mathcal{Z}_G and $G\text{-Cov}$.

4.1. Triple covers and the locus where $\langle -, - \rangle: \det \mathcal{F} \rightarrow \mathcal{L}$ is an isomorphism. The aim of this section is to study the following locus.

Definition 4.1. We denote by \mathcal{U}_ω the (open) substack of $\chi \in G\text{-Cov}$ such that $\langle -, - \rangle_\chi: \det \mathcal{F}_\chi \rightarrow \mathcal{L}_\chi$ is an isomorphism.

Definition 4.2. Define \mathcal{C}_3 as the stack whose objects are pairs (\mathcal{F}, δ) where \mathcal{F} is a locally free sheaf of rank 2 and δ is a map

$$\delta: \text{Sym}^3 \mathcal{F} \rightarrow \det \mathcal{F}$$

Denote by Cov_3 the stack of degree 3 covers, or, equivalently, the stack of locally free sheaves of algebras of rank 3.

Notice that \mathcal{C}_3 is a smooth stack over \mathcal{R} because it is a vector bundle over B GL_2 , which is the stack of rank 2 locally free sheaves. We are going to define a functor $\mathcal{C}_3 \rightarrow \mathcal{U}_\omega$ and prove that it is an isomorphism. This also explains the reason of the section name: it is a classical result (see [Mir85, BV12, Par89]) that, over \mathcal{R}_3 , the stack \mathcal{C}_3 is isomorphic to the stack Cov_3 . We will show that, in this case, the map $\text{Cov}_3 \simeq \mathcal{C}_3 \rightarrow \mathcal{U}_\omega$ is a section of the map $G\text{-Cov} \rightarrow \text{Cov}_3$, obtained taking invariants by $\sigma \in \mathbb{Z}/2\mathbb{Z} \subseteq G$.

In the appendix B.1 we discussed several constructions obtained from an object $(\mathcal{F}, \delta) \in \mathcal{C}_3$. By B.10 \mathcal{C}_3 can also be described as the stack of pairs (\mathcal{F}, β) where \mathcal{F} is a locally free sheaf of rank 2 and $\beta: \text{Sym}^2 \mathcal{F} \rightarrow \mathcal{F}$ is a map such that $\text{tr } \beta = 0$. We follow notation from B.11, in particular the definition of $\delta_\beta, \beta_\delta, \eta_\delta, \alpha_\delta, m_\delta$.

Theorem 4.3. *The maps of stacks*

$$\begin{array}{ccc} (\mathcal{F}, \delta) & \xrightarrow{\quad} & (\det \mathcal{F}, \mathcal{F}, \alpha_\delta, \beta_\delta, \text{id}_{\det \mathcal{F}}) \\ \mathcal{C}_3 & \xrightarrow{\quad \Lambda \quad} & \mathcal{U}_\omega \\ (\mathcal{F}, \delta_\beta) & \xleftarrow{\quad} & (\mathcal{L}, \mathcal{F}, \alpha, \beta, \langle -, - \rangle) \end{array}$$

are well defined and they are quasi-inverses of each other. In particular \mathcal{U}_ω is a smooth open substack of $G\text{-Cov}$. Moreover, over \mathcal{R}_3 , the composition $\text{Cov}_3 \simeq \mathcal{C}_3 \xrightarrow{\Lambda} G\text{-Cov}$ is a section of the map $G\text{-Cov} \rightarrow \text{Cov}_3$ obtained by taking invariants by $\sigma \in \mathbb{Z}/2\mathbb{Z}$ and the same result hold if we replace $G\text{-Cov}$ by $S_3\text{-Cov}$.

We will prove this theorem after collecting some preliminary remarks.

Remark 4.4. By [B.12](#), if $\chi \in G\text{-Cov}$ and $\text{tr } \beta_\chi = 0$, so that $\beta_\chi = \beta_\delta$ (with $\delta = \delta_{\beta_\chi}$), then, by [\(2.4\)](#) and [\(2.15\)](#), we have

$$(4.1) \quad \eta_\delta = 2(-, -)_\chi$$

We now want to show the relationship between \mathcal{C}_3 and Cov_3 . The reader can refer to [[Mir85](#), [BV12](#)] for details and proofs.

Remark 4.5. If $\Phi = (\mathcal{F}, \delta) \in \mathcal{C}_3$ and we set $\mathcal{A}_\Phi = \mathcal{O}_T \oplus \mathcal{F}$, we can endow \mathcal{A}_Φ by a structure of \mathcal{O}_T -algebras given by

$$\text{Sym}^2 \mathcal{F} \xrightarrow{\eta_\delta + \beta_\delta} \mathcal{A}_\Phi$$

This association defines a map of stacks $\mathcal{C}_3 \rightarrow \text{Cov}_3$. This map is an isomorphism if 3 is inverted in the base scheme. Indeed, over \mathcal{R}_3 , if $\mathcal{A} \in \text{Cov}_3$, the trace map $\text{tr}_{\mathcal{A}/\mathcal{O}_T}: \mathcal{A} \rightarrow \mathcal{O}_T$ is surjective and we can write $\mathcal{A} = \mathcal{O}_S \oplus \mathcal{F}$, where $\mathcal{F} = \ker \text{tr}_{\mathcal{A}}$. The multiplication of \mathcal{A} induces a map $\beta: \text{Sym}^2 \mathcal{F} \rightarrow \mathcal{F}$ such that $\text{tr } \beta = 0$ and therefore a $\delta: \text{Sym}^3 \mathcal{F} \rightarrow \det \mathcal{F}$ such that $\beta_\delta = \beta$.

Now let $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -, - \rangle) \in G\text{-Cov}$. It's easy to see that

$$\mathcal{A}_\chi^\sigma = \{a \oplus 0 \oplus x_1 \oplus x_2 \mid a \in \mathcal{O}_T, x_1 = x_2 \in \mathcal{F}\}$$

where $\sigma \in \mathbb{Z}/2\mathbb{Z} \subseteq G$. The map

$$\begin{array}{ccc} \mathcal{O}_T \oplus \mathcal{F} & \longrightarrow & \mathcal{A}_\chi^\sigma \\ a \oplus x & \longmapsto & a \oplus 0 \oplus x \oplus x \end{array}$$

is an isomorphism of \mathcal{O}_S -modules and the induced algebra structure on $\mathcal{O}_T \oplus \mathcal{F}$ is given by

$$\beta: \text{Sym}^2 \mathcal{F} \rightarrow \mathcal{F} \text{ and } 2(-, -): \text{Sym}^2 \mathcal{F} \rightarrow \mathcal{O}_T$$

Moreover $(\text{tr}_{\mathcal{A}_\chi^\sigma})|_{\mathcal{O}} = 3\text{id}$ and $(\text{tr}_{\mathcal{A}_\chi^\sigma})|_{\mathcal{F}} = \text{tr } \beta$. Over \mathcal{R}_3 this means $\mathcal{F} = \ker \text{tr}_{\mathcal{A}_\chi^\sigma}$ if and only if $\text{tr } \beta = 0$. In general we obtain a map of stacks $\{\text{tr } \beta = 0\} = (G\text{-Cov})^{\text{red}} \rightarrow \mathcal{C}_3$ (see [3.5](#)) and, by [4.4](#), $(G\text{-Cov})^{\text{red}} \rightarrow \mathcal{C}_3 \rightarrow \text{Cov}_3$ consists in taking invariants by σ .

Remark 4.6. Theorem [1.5](#) tells us that, over \mathcal{R}_3 , the isomorphism $G\text{-Cov} \simeq S_3\text{-Cov}$ preserves the quotient by $\sigma \in \mathbb{Z}/2\mathbb{Z}$, that is we have a commutative diagram

$$\begin{array}{ccc} G\text{-Cov} & \xrightarrow{\simeq} & S_3\text{-Cov} \\ \searrow & & \swarrow \\ & \text{Cov}_3 & \end{array} \quad \begin{array}{cc} X & \mathcal{A} \\ \downarrow & \downarrow \\ X/\sigma & \mathcal{A}^\sigma \end{array}$$

Proof of Theorem [4.3](#). We need to prove that Λ is well defined. Let $\Phi = (\mathcal{F}, \delta) \in \mathcal{C}_3$. We have that $\chi = \Lambda(\Phi) \in \mathcal{Y}$ and we have to prove that χ satisfies the conditions of [2.11](#). We can therefore work locally and fix a basis y, z of \mathcal{F} . By [\(B.3\)](#), the parameters associated to χ (see [2.12](#)) are

$$a, b, c, d = -a, e, f = -c, \omega = 1, A = -D = (ac + be)/2, B = c^2 - ae, C = -a^2 - bc$$

It is easy to check that all the conditions in [2.11](#) are satisfied. So $\Lambda(\Phi) \in G\text{-Cov}$ and, by definition, $\Lambda(\Phi) \in \mathcal{U}_\omega$.

Following notation from 3.1 we have Cartesian diagrams

$$\begin{array}{ccccccc} \mathrm{Spec} S & \longrightarrow & \mathrm{Spec} P_\omega & \longrightarrow & \mathrm{Spec} P & \longrightarrow & \mathrm{Spec} \mathcal{R} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}_3 & \xrightarrow{\Lambda} & \mathcal{U}_\omega & \longrightarrow & G\text{-Cov} & \longrightarrow & \mathrm{BGL}_2 \times \mathrm{B}\mathbb{G}_m \end{array}$$

By definition of \mathcal{C}_3 one have $S = \mathcal{R}[a, b, c, e, \omega]_\omega$ and $P \rightarrow S$ is defined as above. In order to conclude that Λ is an equivalence it is enough to notice that $P_\omega \rightarrow S$ is an isomorphism. This holds because, if ω is invertible, then $a = -d$, $c = -f$, $A = -D$ and B and C are functions of the a, b, c, e . By a direct check, the only two non trivial relations missing are automatically satisfied.

Now assume we are over \mathcal{R}_3 . The map $G\text{-Cov} \rightarrow \mathrm{Cov}_3 \simeq \mathcal{C}_3$ extends the map $\mathcal{U}_\omega \rightarrow \mathcal{C}_3$ defined in the statement. Therefore $\mathrm{Cov}_3 \simeq \mathcal{C}_3 \rightarrow \mathcal{U}_\omega \subseteq G\text{-Cov}$ is a section of such map. The claim about S_3 follows from 4.6. \square

Corollary 4.7. *Set $\mathcal{F} = \mathcal{R}^2$ with basis e_1, e_2 and consider $\delta: \mathrm{Sym}^3 \mathcal{F} \rightarrow \det \mathcal{F}$ given by $\delta(e_2^3) = -\delta(e_1^3) = 1$ and $\delta(e_1 e_2^2) = \delta(e_1^2 e_2) = 0$. Then*

$$G \simeq \underline{\mathrm{Aut}}_{\mathcal{C}_3}(\mathcal{F}, \delta)$$

Over \mathcal{R}_3 , the map $\mathrm{B}G \rightarrow \mathrm{Cov}_3$ obtained by taking invariants by $\sigma \in \mathbb{Z}/2\mathbb{Z}$, is an equivalence onto the locus Et_3 of étale degree 3 covers. Moreover

$$G \simeq \underline{\mathrm{Aut}}_{\mathrm{Cov}_3}(\mathcal{R}_3[t]/(t^3 - 1))$$

Proof. By [Ton17a, Thm A], the trivial G -torsor $G \rightarrow \mathrm{Spec} \mathcal{R}$ is associated with the forgetful functor $\Omega: \mathrm{Loc}^G \mathcal{R} \rightarrow \mathrm{Loc} \mathcal{R}$. In particular, taking into account section 2.1, the sequence $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -, - \rangle) \in G\text{-Cov}(\mathcal{R})$ associated with Ω (and the trivial G -torsor) is given by

$$\mathcal{L} = A, \mathcal{F} = V; \quad \begin{array}{l} \alpha(1_A \otimes v_1) = -v_1 \quad ; \quad \beta(v_1^2) = v_2, \quad \beta(v_1 v_2) = 0, \quad \beta(v_2^2) = v_1 \\ \alpha(1_A \otimes v_2) = v_2 \quad ; \quad \langle v_1, v_2 \rangle = (1/2)1_A, \quad m(1_A \otimes 1_A) = 1 \end{array}$$

In particular $\chi \in \mathcal{U}_\omega$ and, by 4.3, the image of χ via the equivalence $\Phi: \mathcal{U}_\omega \rightarrow \mathcal{C}_3$ is $(\mathcal{F}, \delta_\beta)$. Moreover $G \simeq \underline{\mathrm{Aut}}_{G\text{-Cov}} \chi \simeq \underline{\mathrm{Aut}}_{\mathcal{C}_3}(\mathcal{F}, \delta_\beta)$ and, by (B.1), $\delta_\beta = \delta$.

Assume now that the base ring is \mathcal{R}_3 , so that $\Phi: \mathcal{U}_\omega \rightarrow \mathcal{C}_3 \simeq \mathrm{Cov}_3$ consists in taking invariants by σ . By 4.5 the algebra $\Phi(\chi)$ associated with χ or (\mathcal{F}, δ) is $S = \mathcal{R}_3 \oplus \mathcal{F}$ with multiplication $2(-, -) \oplus \beta: \mathrm{Sym}^2 \mathcal{F} \rightarrow \mathcal{R}_3 \oplus \mathcal{F}$. A direct computation shows that $v_1^2 = v_2$ and $v_1^3 = 1$. It follows that $S = \mathcal{R}_3[t]/(t^3 - 1)$ and, in particular, the last isomorphism. Moreover $\Phi|_{\mathrm{B}G}: \mathrm{B}G \rightarrow \mathrm{Cov}_3$ is an equivalence onto the substack E of Cov_3 of objects which are locally isomorphic to $\Phi(\chi) = S$. Since $\mathrm{Spec} S \rightarrow \mathrm{Spec} \mathcal{R}_3$ is an étale cover of degree 3, it follows that $E = \mathrm{Et}_3$. \square

4.2. The locus where $\alpha: \mathcal{L} \otimes \mathcal{F} \rightarrow \mathcal{F}$ is nowhere a multiple of the identity.

Definition 4.8. We denote by \mathcal{U}_α the full substack of $\chi \in G\text{-Cov}$ such that $\alpha_\chi: \mathcal{L}_\chi \otimes \mathcal{F}_\chi \rightarrow \mathcal{F}_\chi$ is nowhere a multiple of the identity, i.e. it is not a multiple of the identity over all geometric points of the base (see also B.4).

Theorem 4.9. *Let $R = \mathcal{R}[m, a, b]$. Then*

$$(R, R^2, \alpha, \beta, \langle -, - \rangle) \text{ where } \alpha = \begin{pmatrix} 0 & m \\ 1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} a & -mb & ma \\ b & -a & mb \end{pmatrix}, \quad \langle -, - \rangle = mb^2 - a^2$$

is an object of $G\text{-Cov}(R)$. The induced map $\mathbb{A}^3 \rightarrow G\text{-Cov}$ is a smooth Zariski epimorphism onto \mathcal{U}_α . In particular \mathcal{U}_α is a smooth open substack of $G\text{-Cov}$.

Before proving this Theorem we need two lemmas.

Lemma 4.10. *Let \mathcal{F} be a locally free sheaf of rank 2, \mathcal{L} be an invertible sheaf, both over a scheme T and $\alpha: \mathcal{L} \otimes \mathcal{F} \rightarrow \mathcal{F}$ be a map. Let also k be a field, $\text{Spec } k \rightarrow T$ be a map and $p \in T$ the induced point. If $\alpha \otimes k$ is not a multiple of the identity, then there exists a Zariski open neighborhood V of p in T and $y \in \mathcal{F}|_V$ such that $\mathcal{L}|_V = \mathcal{O}_V$ and $y, \alpha(t \otimes y)$ is a basis of $\mathcal{F}|_V$.*

Proof. By Nakayama's lemma we can assume $T = \text{Spec } k$. By contradiction assume that a basis as in the statement does not exist. It is easy to deduce that any vector of \mathcal{F} is an eigenvector for α . By standard linear algebra we can conclude that α is a multiple of the identity. \square

Lemma 4.11. *Let $\chi = (\mathcal{L}, \mathcal{F}, \alpha, \beta, \langle -, - \rangle) \in \mathcal{Y}$ and $y \in \mathcal{F}$ be such that $\mathcal{L} = \mathcal{O}_T$ and $y, z = \alpha(y)$ is a basis of \mathcal{F} . Then $\chi \in G\text{-Cov}$ if and only if the associated parameters (see 2.12) of χ with respect to the basis y, z are*

$$a, b, c = -mb, d = -a, e = ma, f = mb, \omega = mb^2 - a^2, A = D = 0, B = m, C = 1$$

In this case $\chi \in \mathcal{U}_\alpha$.

Proof. First of all note that, if the associated parameters of χ are as above, then they satisfy equations (2.15). Therefore $\chi \in G\text{-Cov}$ and, by definition, α is nowhere a multiple of the identity, i.e. $\chi \in \mathcal{U}_\alpha$. Consider now the converse implication and denote by $a, b, c, d, e, f, \omega, A, B, C, D, m$ the parameters associated with $\chi \in G\text{-Cov}$ with respect to the basis y, z of \mathcal{F} . In particular equations (2.15) hold true. By definition of y, z we have $A = 0, C = 1$ and therefore

$$\begin{aligned} C(A + D) = 0 &\implies D = 0, m = (A^2 + D^2)/2 + BC = B \\ b(A + D) + C(a + d) = d(A + D) + C(c + f) = 0 &\implies d = -a, f = -c \\ (2aA + bB + cC) = (2cA + dB + eC) = 0 &\implies c = -mb, e = ma \\ a^2 + bc = -\omega C &\implies \omega = mb^2 - a^2 \end{aligned}$$

\square

Proof. (of theorem 4.9). By 4.10 and 4.11, \mathcal{U}_α is an open substack of $G\text{-Cov}$, $\chi \in \mathcal{U}_\alpha(R)$ and the induced map $\pi: \mathbb{A}^3 \rightarrow \mathcal{U}_\alpha$ is a Zariski epimorphism.

It remains to prove that π is smooth. The scheme \mathbb{A}^3 represents the functor $(\text{Sch}/\mathcal{R})^{\text{op}} \rightarrow (\text{Sets})$ associating with a scheme T the setoid of tuples (χ, t, y) where $\chi = (\mathcal{L}, \mathcal{F}, \alpha, \beta, \langle -, - \rangle) \in G\text{-Cov}(T)$, $t \in \mathcal{L}$ is a generator and $y \in \mathcal{F}$ is an element such that $y, \alpha(t \otimes y)$ is a basis for \mathcal{F} . In particular, given a map $\chi = (\mathcal{L}, \mathcal{F}, \alpha, \beta, \langle -, - \rangle): T \rightarrow \mathcal{U}_\alpha$ the fiber product $Z = T \times_{\mathcal{U}_\alpha} \mathbb{A}^3$ represents the functor $(\text{Sch}/T)^{\text{op}} \rightarrow (\text{Sets})$ of pairs (t, y) where $t \in \mathcal{L}$ is a generator and $y, \alpha(t \otimes y)$ is a basis of \mathcal{F} .

The smoothness of $Z \rightarrow T$ is local on T , therefore we can assume $\chi = \pi(m, a, b)$ for $m, a, b \in \mathcal{O}_T$. If $y = (u, v) \in \mathcal{F} = \mathcal{O}_T^2$ then $\alpha(y) = (mv, u)$ and $y, \alpha(y)$ is a basis of \mathcal{F} if and only if $u^2 - mv^2$ is invertible. This observation allows us to conclude that

$$Z = \text{Spec}(\mathcal{O}_T[X, Y, W]_{X(Y^2 - mW^2)}) \subseteq \mathbb{A}_T^3$$

is open, hence smooth. \square

4.3. The locus where $\beta: \text{Sym}^2 \mathcal{F} \rightarrow \mathcal{F}$ is nowhere zero. Given an object $\chi \in G\text{-Cov}$ or a morphism $\beta: \text{Sym}^2 \mathcal{F} \rightarrow \mathcal{F}$ (where \mathcal{F} is a rank 2 locally free sheaf) we define the map

$$d_\beta: \mathcal{F} \rightarrow \det \mathcal{F}, x \mapsto x \wedge \beta(x^2)$$

Notice that this is not a map of quasi-coherent sheaves because it is not \mathcal{O} -linear, it is just a map of sheaves of sets. We remark that its formation commutes with arbitrary base changes of the base.

Definition 4.12. We define \mathcal{U}_β as the full substack of objects $\chi \in G\text{-Cov}$ such that $d_{\beta_\chi} : \mathcal{F}_\chi \rightarrow \det \mathcal{F}_\chi$ is nowhere zero, i.e. d_{β_χ} is not zero on all geometric points of the base.

Theorem 4.13. Let $R = \mathcal{R}[\omega, A, C]$. Then

$$(R, R^2, \alpha, \beta, \langle -, - \rangle) \text{ where } \alpha = \begin{pmatrix} A & \omega C^2 \\ C & -A \end{pmatrix}, \beta = \begin{pmatrix} 0 & -\omega C & 2\omega A \\ 1 & 0 & \omega C \end{pmatrix}, \langle -, - \rangle = \omega$$

is an object of $G\text{-Cov}(R)$. The associated map $\pi : \mathbb{A}^3 \rightarrow G\text{-Cov}$ is smooth and its image is \mathcal{U}_β , which is therefore a smooth open substack of $G\text{-Cov}$.

Over \mathcal{R}_3 , \mathcal{U}_β coincides with the full substack of $\chi \in G\text{-Cov}$ such that β_χ is nowhere zero.

We prove the above result at the end of the section.

Lemma 4.14. Let T be an \mathcal{R} -scheme, \mathcal{F} a free sheaf of rank 2 and

$$\beta = \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix} : \text{Sym}^2 \mathcal{F} \rightarrow \mathcal{F}$$

a morphism. Then the locus U where $d_\beta : \mathcal{F} \rightarrow \det \mathcal{F}$ is nowhere zero is open and its complement is defined by the ideal $(b, e, 2d - a, f - 2c)$. In particular if T is an \mathcal{R}_3 -scheme and $\text{tr } \beta = 0$ then d_β is nowhere zero if and only if β is nowhere zero.

Moreover for any $u \in U$ there exists an open neighborhood V of u inside U and $y \in \mathcal{F}(V)$ such that $y, \beta(y^2)$ is a basis of $\mathcal{F}|_V$.

Proof. Set $\mathcal{F} = \mathcal{O}_T^2$ with basis e_1, e_2 . Given $y = ue_1 + ve_2$ a direct computation shows that

$$d_\beta(y) = y \wedge \beta(y^2) = (u^3b + u^2v(2d - a) + uv^2(f - 2c) - v^3e)e_1 \wedge e_2$$

Choosing $y = e_1, e_2, e_1 + e_2, e_1 - e_2$ we see that

$$d_\beta = 0 \iff b = e = 2d - a = f - 2c = 0$$

which proves the first claims. For the last one, by Nakayama there exist a neighborhood $u \in V$, $y \in \mathcal{F}$ and $q \in \mathcal{O}_V$ not zero in u such that $d_\beta(y) = q(e_1 \wedge e_2) \in \det \mathcal{F}|_V$. Inverting q in V we find the desired neighborhood. \square

Lemma 4.15. Let $\chi = (\mathcal{L}, \mathcal{F}, \alpha, \beta, \langle -, - \rangle) \in \mathcal{Y}$ and $y \in \mathcal{F}$ be such that $\mathcal{L} = \mathcal{O}_T$ and $y, z = \beta(y^2)$ is a basis of \mathcal{F} . Then $\chi \in G\text{-Cov}$ if and only if the associated parameters (see 2.12) of χ with respect to the basis y, z are

$$a = 0, b = 1, c = -\omega C, d = 0, e = 2\omega A, f = \omega C, \omega, A, B = \omega C^2, C, D = -A$$

In this case $\chi \in \mathcal{U}_\beta$.

Proof. First of all, it is easy to check that, if the associated parameters of χ are the ones listed in the statement, then they satisfy equations (2.15). Therefore $\chi \in G\text{-Cov}$ and, since $\beta(y^2) \neq 0$ after all base changes, $\chi \in \mathcal{U}_\beta$.

Assume now that $\chi \in G\text{-Cov}$. By definition of the basis y, z , we have $a = 0$ and $b = 1$. Using relations (2.15), we also have

$$b(a + d) = a(a + d) + b(c + f) = 0 \implies d = -a = 0, f = -c$$

$$b(A + D) + C(a + d) = 0 \implies D = -A$$

$$a^2 + bc = -\omega C, ac + be = 2\omega A \implies c = -\omega C, e = 2\omega A$$

$$2aA + bB + cC = 0 \implies B = \omega C^2$$

\square

Proof of Theorem 4.13. From 4.14 and 4.15 we see that $(R, R^2, \alpha, \beta, \langle -, - \rangle) \in \mathcal{U}_\beta(R)$, $\pi: \mathbb{A}^3 \rightarrow \mathcal{U}_\beta$ is a Zariski epimorphism and \mathcal{U}_β is open. The last claim follows from 4.14 and the fact that if $\chi \in G\text{-Cov}(k)$ for a field k then $\text{tr } \beta_\chi = 0$ by 3.5.

It remains to prove that π is smooth. By 4.15 the scheme \mathbb{A}^3 represents the functor $(\text{Sch}/\mathcal{R})^{\text{op}} \rightarrow (\text{Sets})$ associating with a scheme T the setoid of tuples (χ, t, y) where $\chi = (\mathcal{L}, \mathcal{F}, \alpha, \beta, \langle -, - \rangle) \in G\text{-Cov}(T)$, $t \in \mathcal{L}$ is a generator and $y \in \mathcal{F}$ is an element such that $y, \beta(y^2)$ is a basis for \mathcal{F} . In particular, given a map $\chi = (\mathcal{L}, \mathcal{F}, \alpha, \beta, \langle -, - \rangle): T \rightarrow \mathcal{U}_\alpha$ the fiber product $Z = T \times_{\mathcal{U}_\alpha} \mathbb{A}^3$ represents the functor $(\text{Sch}/T)^{\text{op}} \rightarrow (\text{Sets})$ of pairs (t, y) where $t \in \mathcal{L}$ is a generator and $y, \beta(y^2)$ is a basis of \mathcal{F} .

The smoothness of $Z \rightarrow T$ can be checked locally on T and therefore we can assume $\mathcal{L} = \mathcal{O}_T$ and $\mathcal{F} = \mathcal{O}_T^2$, so that $Z \subseteq \mathbb{A}_T^3$. If $y = (u, v) \in \mathcal{F}$ then $\beta(y^2) = (f(u, v), g(u, v))$ for some $f, g \in \mathcal{O}_T[X, Y]$ and $y, \beta(y^2)$ is a basis of \mathcal{F} if and only if $ug(u, v) - vf(u, v)$ is invertible in the base. This implies that

$$Z = \text{Spec } \mathcal{O}_T[W, X, Y]_{W(Xg - Yf)} \subseteq \mathbb{A}_T^3$$

is open, hence smooth. \square

4.4. The stack of torsors $\mathbf{B}(\mu_3 \times \mathbb{Z}/2\mathbb{Z})$. In this section we want to describe the stack $\mathbf{B}G$ of G -torsors.

Definition 4.16. Given $\chi = (\mathcal{F}, \delta) \in \mathcal{C}_3$ (see 4.3) (resp. $\chi = (\mathcal{L}, \mathcal{F}, \alpha, \beta, \langle -, - \rangle) \in G\text{-Cov}$) we define the discriminant map $\Delta_\chi: (\det \mathcal{F})^2 \rightarrow \mathcal{O}_T$ as the map obtained from $\mathcal{F} \otimes \mathcal{F} \rightarrow \text{Sym}^2 \mathcal{F} \xrightarrow{\eta_\delta} \mathcal{O}_T$ (resp. $\mathcal{F} \otimes \mathcal{F} \rightarrow \text{Sym}^2 \mathcal{F} \xrightarrow{\langle -, - \rangle_\chi} \mathcal{O}_T$) as in B.7.

Remark 4.17. For $\chi = (\mathcal{L}, \mathcal{F}, \alpha, \beta, \langle -, - \rangle) \in G\text{-Cov}$ the map Δ_χ coincides with

$$(\det \mathcal{F})^2 \xrightarrow{\langle -, - \rangle^{\otimes 2}} \mathcal{L}^2 \xrightarrow{-m_\chi} \mathcal{O}_T$$

Indeed locally one has $\Delta_\chi = (y, y)(z, z) - (y, z)^2 = -BC\omega^2 - A^2\omega^2 = -\omega^2 m$.

Moreover, if $\text{tr } \beta = 0$, then $\Delta_{(\mathcal{F}, \delta_\beta)} = 4\Delta_\chi$ thanks to 4.1.

Remark 4.18. Let $\Phi = (\mathcal{F}, \delta) \in \mathcal{C}_3$ and $\mathcal{A}_\Phi = \mathcal{O}_T \oplus \mathcal{F}$ be the algebra associated with Φ (see 4.5). By B.8 the discriminant of \mathcal{A} is $\Delta_{\mathcal{A}_\Phi} = 3^3 \Delta_\Phi$ and, over \mathcal{R}_3 , the algebra \mathcal{A}_Φ is étale over T if and only if Δ_Φ is an isomorphism.

Theorem 4.19. An object $\chi = (\mathcal{L}, \mathcal{F}, m, \alpha, \beta, \langle -, - \rangle) \in G\text{-Cov}$ corresponds to a G -torsor if and only if the maps

$$m: \mathcal{L}^2 \rightarrow \mathcal{O}_T, \langle -, - \rangle: \det \mathcal{F} \rightarrow \mathcal{L}$$

are isomorphisms, or, equivalently, $\Delta_\chi: (\det \mathcal{F})^2 \rightarrow \mathcal{O}_T$ is an isomorphism. In this case the maps

$$\alpha: \mathcal{L} \otimes \mathcal{F} \rightarrow \mathcal{F} \text{ and } (-, -) \oplus \langle -, - \rangle \oplus \beta: \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{O} \oplus \mathcal{L} \oplus \mathcal{F}$$

are isomorphisms and $\text{tr } \beta = \text{tr } \alpha = 0$. Moreover $\mathbf{B}G \subseteq \mathcal{U}_\omega, \mathcal{U}_\alpha, \mathcal{U}_\beta$.

Finally the map $\Lambda: \mathcal{C}_3 \rightarrow G\text{-Cov}$ of Theorem 4.3 restricts to an equivalence between the full substack of \mathcal{C}_3 of objects Φ such that Δ_Φ is an isomorphism and $\mathbf{B}G$.

Proof. Let Γ be the functor associated with χ . By [Ton17a, Thm A] χ corresponds to a G -torsor if and only if Γ is strong monoidal, that is all maps $\Gamma_U \otimes \Gamma_W \rightarrow \Gamma_{U \otimes W}$ are isomorphisms for $U, W \in \text{Loc}^G \mathcal{R}$. Equivalently: the maps

$$\Gamma_A \otimes \Gamma_A \rightarrow \Gamma_{A \otimes A} \simeq \Gamma_{\mathcal{R}}, \Gamma_A \otimes \Gamma_V \rightarrow \Gamma_{A \otimes V} \simeq \Gamma_V, \Gamma_V \otimes \Gamma_V \rightarrow \Gamma_{V \otimes V} \simeq \Gamma_{\mathcal{R}} \oplus \Gamma_A \oplus \Gamma_V$$

which are exactly m, α and $(-, -) \oplus \langle -, - \rangle \oplus \beta$, are isomorphisms. If this is the case then $\langle -, - \rangle: \det \mathcal{F} \rightarrow \mathcal{L}$ is surjective, hence an isomorphism. In particular $\mathbf{B}G \subseteq \mathcal{U}_\omega, \mathcal{U}_\beta$, $\text{tr } \beta = 0$, and, thanks to 4.17, the last claim follows from the first ones.

Conversely assume that m and $\langle -, - \rangle$ are isomorphisms. The map α is an isomorphism because $\alpha^2 = \text{id}$ locally. Moreover $\chi \in \mathcal{U}_\omega$ implies that $\text{tr } \alpha = 0$, as one can check locally. If by contradiction $\chi \notin \mathcal{U}_\alpha$ then, on some geometric point, we would have that α is a multiple of the identity and therefore $\alpha = (\text{tr } \alpha/2)\text{id} = 0$, which is not the case. So $\chi \in \mathcal{U}_\alpha$.

It remains to prove that $\chi \in \text{BG}$. Since this is a local statement, we can assume $\chi = \pi(m, a, b)$, where $\pi: \mathbb{A}^3 \rightarrow \mathcal{U}_\alpha$ is the map defined in 4.9. We have to prove that $(-, -) \oplus \langle -, - \rangle \oplus \beta: \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{O} \oplus \mathcal{L} \oplus \mathcal{F}$ is an isomorphism. If M is its attached matrix we have to show that $\det M$ is invertible. We have

$$M = \begin{pmatrix} -\omega & 0 & 0 & m\omega \\ 0 & -\omega & \omega & 0 \\ a & -mb & -mb & ma \\ b & -a & -a & mb \end{pmatrix} \implies N = \begin{pmatrix} -\omega & 0 & 0 & 2m\omega \\ 0 & -\omega & 2\omega & 0 \\ a & -mb & 0 & 0 \\ b & -a & 0 & 0 \end{pmatrix}$$

The matrix N is obtained replacing the third and fourth columns of M by $M^3 - M^2$ and $M^4 - mM^1$ respectively, where M^j denotes the j -th column. Thus

$$\det M = \det N = -2m\omega \cdot 2\omega \cdot (mb^2 - a^2) = -4m\omega^3 \text{ is invertible}$$

□

4.5. The main irreducible component $\mathcal{Z}_{(\mu_3 \times \mathbb{Z}/2\mathbb{Z})}$. In this subsection we want to give a more precise description of the irreducible component \mathcal{Z}_G of $G\text{-Cov}$ and, because of 1.5, of $\mathcal{Z}_{S_3} \subseteq S_3\text{-Cov}$ over \mathcal{R}_3 (see 1.3). We are going to use results and notation from section B.2. In particular we use notation B.16.

Define the stack \mathcal{Z} whose objects are tuples $(\mathcal{M}, \mathcal{F}, \delta, \zeta, \omega)$ where \mathcal{M} is an invertible sheaf, \mathcal{F} is a locally free sheaf of rank 2, ω is a section of \mathcal{M} and δ, ζ are maps

$$\delta: \text{Sym}^3 \mathcal{F} \longrightarrow \det \mathcal{F}, \quad \zeta: (\det \mathcal{F})^2 \otimes \mathcal{M} \longrightarrow \text{Sym}^2 \mathcal{F}$$

satisfying the following conditions:

- 1) the composition

$$(4.2) \quad (\det \mathcal{F})^2 \otimes \mathcal{M} \otimes \mathcal{F} \xrightarrow{\zeta \otimes \text{id}} \text{Sym}^2 \mathcal{F} \otimes \mathcal{F} \longrightarrow \text{Sym}^3 \mathcal{F} \xrightarrow{\delta} \det \mathcal{F}$$

is zero;

- 2) the composition

$$(4.3) \quad \text{Sym}^2 \mathcal{F} \xrightarrow{\zeta} \mathcal{M}^{-1} \xrightarrow{\omega^\vee} \mathcal{O}$$

coincides with η_δ (see B.16 and (B.2)).

We define a functor

$$\mathcal{Z} \rightarrow \mathcal{Y}, \quad \Omega = (\mathcal{M}, \mathcal{F}, \delta, \zeta, \omega) \longmapsto (\mathcal{L}_\Omega, \mathcal{F}_\Omega, \alpha_\Omega, \beta_\Omega, \langle -, - \rangle_\Omega)$$

setting $\mathcal{L}_\Omega = \mathcal{M} \otimes \det \mathcal{F}$, $\mathcal{F}_\Omega = \mathcal{F}$, $\alpha_\Omega: \mathcal{L}_\Omega \otimes \mathcal{F} \rightarrow \mathcal{F}$ the trace 0 map obtained from ζ via B.14, $\beta_\Omega = \beta_\delta: \text{Sym}^2 \mathcal{F} \rightarrow \mathcal{F}$ (see (B.1)) and finally $\langle -, - \rangle_\Omega = \omega \otimes \text{id}_{\det \mathcal{F}}: \det \mathcal{F} \rightarrow \mathcal{M} \otimes \det \mathcal{F} = \mathcal{L}_\Omega$.

Remembering the notation introduced in B.9, we want to prove the following Theorem.

Theorem 4.20. *We have*

$$\mathcal{Z}_G - \{0\} = \mathcal{U}_\omega \cup \mathcal{U}_\alpha \cup \mathcal{U}_\beta$$

(see 3.3) and it is the smooth locus of $\mathcal{Z}_G \rightarrow \text{Spec } \mathcal{R}$.

Moreover we have an equivalence of stacks.

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\quad\quad\quad} & \mathcal{Z}_G \\ \Omega = (\mathcal{M}, \mathcal{F}, \delta, \zeta, \omega) & \longmapsto & (\mathcal{L}_\Omega, \mathcal{F}_\Omega, \alpha_\Omega, \beta_\Omega, \langle -, - \rangle_\Omega) \end{array}$$

Corollary 4.21. *The scheme $X = \text{Proj}(\mathcal{R}[a, b, c, e, A, B, C, \omega]/I) \rightarrow \text{Spec } \mathcal{R}$, where*

$$I = (2aA + bB + cC, 2cA - aB + eC, 2\omega A - (ac + be), \omega B - (c^2 - ae), \omega C + a^2 + bc)$$

is smooth and projective and its geometric fibers are non degenerated surfaces in \mathbb{P}^7 . Moreover $(\mathcal{Z}_G - \{0\}) \simeq [X/\text{GL}_2]$.

Proof. The global claims follows from 3.13 and 4.20. For the geometry of the fibers, we base change to a geometric point of the base. Since $f: X \rightarrow (\mathcal{Z}_G - \{0\})$ has relative dimension 2 and BG is a 0-dimensional open substack of $\mathcal{Z}_G - \{0\}$, it follows that $\dim X = \dim f^{-1}(BG) = 2$. \square

We will prove Theorem 4.20 after the following lemma.

Lemma 4.22. *Let $\chi = (\mathcal{L}, \mathcal{F}, \alpha, \beta, \langle -, - \rangle) \in \mathcal{Y}(T)$ be such that $\text{tr } \alpha = \text{tr } \beta = 0$ and set $\mathcal{M} = \mathcal{L} \otimes \det \mathcal{F}^{-1}$. Let also $\zeta: \mathcal{M} \otimes (\det \mathcal{F})^2 \rightarrow \text{Sym}^2 \mathcal{F}$ be the map associated with α via B.14 and $\delta = \delta_\beta: \text{Sym}^3 \mathcal{F} \rightarrow \det \mathcal{F}$ (see (B.1)). If $\mathcal{L} = \mathcal{O}_T$, y, z is a basis of \mathcal{F} and we use notation from 2.12, we have equivalences*

$$\begin{aligned} \text{the map (4.2) is zero} &\iff \beta \circ \zeta = 0 \iff \begin{cases} 2aA + bB + cC = 0 \\ 2cA + eC - aB = 0 \end{cases} \\ \text{the map (4.3) coincides with } \eta_\delta &\iff \begin{cases} a^2 + bc = -\omega C \\ ac + be = 2\omega A \\ c^2 - ae = B\omega \end{cases} \end{aligned}$$

Proof. The conditions $\text{tr } \alpha = \text{tr } \beta = 0$ means that $a+d = c+f = A+D = 0$. We have expressions

$$\zeta = By^2 - 2Ayz - Cz^2, \quad \check{\zeta} = -2C(y^2)^* + 2A(yz)^* + 2B(z^2)^*$$

thanks to (B.5) and (B.6). In particular

$$\beta(\zeta) = (aB - 2cA - eC)y + (2aA + bB + cC)z$$

By definition of δ_β , the composition 4.2 is $\mathcal{F} \ni x \mapsto x \wedge \beta(\zeta) \in \det \mathcal{F}$. The first equivalence follows from expressions

$$y \wedge \beta(\zeta) = (2aA + bB + cC)y \wedge z \text{ and } z \wedge \beta(\zeta) = -(aB - 2cA - eC)y \wedge z$$

The second one instead follows from the expression of η_δ given in B.4 and the fact that the map 4.3 is just $\omega\check{\zeta}$. \square

Proof of Theorem 4.20. From 4.22 we see that $\mathcal{Z} \rightarrow G\text{-Cov}$ is well defined and is an equivalence onto the locus $\{\text{tr } \alpha = \text{tr } \beta = 0\}$, which coincides with \mathcal{Z}_G thanks to 3.5.

For the first claim we use Theorems 3.5, 4.3, 4.9 and 4.13. They tell us that $\mathcal{U}_\omega \cup \mathcal{U}_\alpha \cup \mathcal{U}_\beta \subseteq \mathcal{Z}_G$ is an open substack and it is smooth and geometrically integral over \mathcal{R} . By 3.17 it remains to show that, topologically, $\mathcal{Z}_G - (\mathcal{U}_\omega \cup \mathcal{U}_\alpha \cup \mathcal{U}_\beta) = \{0\}$. If k is an algebraically closed field and $\chi \in \mathcal{Z}_G(k)$ is represented by local parameters as usual then $\chi \notin (\mathcal{U}_\omega \cup \mathcal{U}_\alpha \cup \mathcal{U}_\beta)$ if and only if $\omega = 0$, $d_\beta = 0$ and $\alpha = \lambda \text{id}$. As $\text{tr } \alpha = 0$ it follows that $\alpha = 0$. By 4.14 the condition $d_\beta = 0$ implies $b = e = 0$. By 2.15 we also have $a^2 = c^2 = 0$ which tells us that $a = c = 0$ and, since $\text{tr } \beta = 0$, $d = f = 0$. In conclusion $\chi \notin (\mathcal{U}_\omega \cup \mathcal{U}_\alpha \cup \mathcal{U}_\beta)$ if and only if $\chi \in \{0\}$. \square

Proof of Theorem 3.16. The first equivalence is 3.13. Moreover $G\text{-Cov}$ is smooth outside $\{0\}$ by 4.20. The result then follows from 3.17. \square

Remark 4.23. Given $\chi \in \mathcal{Z}_G$ and denoted by $(\mathcal{M}, \mathcal{F}, \delta, \zeta, \omega)$ the corresponding object via 4.20, one can check locally that the composition

$$\mathcal{M}^2 \otimes (\det \mathcal{F})^2 \xrightarrow{\text{id}_{\mathcal{M}} \otimes \zeta} \mathcal{M} \otimes \text{Sym}^2 \mathcal{F} \xrightarrow{\text{id}_{\mathcal{M}} \otimes \check{\zeta}} \mathcal{M} \otimes \mathcal{M}^{-1} \simeq \mathcal{O}_T$$

is $-4m_\chi$, where as usual $m_\chi: \mathcal{L}_\chi^2 \simeq \mathcal{M}^2 \otimes (\det \mathcal{F})^2 \rightarrow \mathcal{O}_S$.

APPENDIX A. BITORSORS.

In this section we recall some properties of bitorsors. One can compare definitions and results with [Gir71, Chapter III, Section 1.5]. Let us fix a site \mathcal{C} and two sheaves of groups $G, H: \mathcal{C}^{\text{op}} \rightarrow (\text{Grps})$.

If $F: \mathcal{C}^{\text{op}} \rightarrow (\text{Sets})$ is a functor and $S \in \mathcal{C}$ we denote by F_S the composition $(\mathcal{C}/S)^{\text{op}} \rightarrow \mathcal{C}^{\text{op}} \rightarrow (\text{Sets})$ and we (improperly) call it the restriction to S . If F is a sheaf (of groups) then F_S is a sheaf (of groups). If $F: (\mathcal{C}/S)^{\text{op}} \rightarrow (\text{Sets})$ is a functor an action of G on F is an action of G_S on F .

In this section

Definition A.1. We denote by $\text{Sh}^G(\mathcal{C})$ (resp. $\text{Sh}_{\mathcal{C}}^G$) the category (resp. fibered category) of sheaves over \mathcal{C} with a right G -action. In particular $\text{Sh}_{\mathcal{C}}^G(S) = \text{Sh}^{G_S}(\mathcal{C}/S)$ for $S \in \mathcal{C}$ and the pullbacks are the restrictions.

The left (resp. right) regular action of G on itself is the left (resp. right) action given by

$$G \times G \longrightarrow G, (g, h) \longrightarrow gh$$

A left (resp. right) G -torsor over \mathcal{C} is a sheaf $P: \mathcal{C}^{\text{op}} \rightarrow (\text{Sets})$ with a left action $G \times P \rightarrow P$ (resp. right action $P \times G \rightarrow P$) such that, for all $T \in \mathcal{C}$, P_T is locally isomorphic to G_T endowed with the left (resp. right) regular action. A left (resp. right) G -torsor over $S \in \mathcal{C}$ is a left (resp. right) G_S -torsor over \mathcal{C}/S .

A (G, H) -biaction on a sheaf $P: \mathcal{C}^{\text{op}} \rightarrow (\text{Sets})$ is a pair $(G \times P \xrightarrow{u} P, P \times H \xrightarrow{v} P)$ where u and v are, respectively, a left G -action and a right H -action on P , such that the following diagram is commutative.

$$\begin{array}{ccc} G \times P \times H & \xrightarrow{u \times \text{id}_H} & P \times H \\ \downarrow \text{id}_G \times v & & \downarrow v \\ G \times P & \xrightarrow{u} & P \end{array}$$

A (G, H) -bitorsor over \mathcal{C} (resp. $S \in \mathcal{C}$) is a sheaf $P: \mathcal{C}^{\text{op}} \rightarrow (\text{Sets})$ (resp. $P: (\mathcal{C}/S)^{\text{op}} \rightarrow (\text{Sets})$) with a (G, H) -biaction for which P is both a left G -torsor and a right H -torsor. Denote by $\text{B}(G, H)$ the fibered category over \mathcal{C} of (G, H) -bitorsors. Notice that a (G, H) torsor over \mathcal{C} corresponds to a section $\mathcal{C} \rightarrow \text{B}(G, H)$.

Remark A.2. The fibered category $\text{B}(G, H)$ is a stack over \mathcal{C} . This is easy to prove directly, using the fact that $\text{Sh}_{\mathcal{C}}$, the fibered category of sheaves of sets over \mathcal{C} , is a stack (see [FGI+05, Part 1, Example 4.11]).

Remark A.3. Given a left G -action $u: G \times P \rightarrow P$ and a right H -action $v: P \times H \rightarrow P$, the pair (u, v) is a (G, H) -biaction if and only if the homomorphism $G \rightarrow \underline{\text{Aut}} P$ induced by u factors through $\underline{\text{Aut}}^H P$, that is if G acts through H -equivariant isomorphisms.

Lemma A.4. *Let P be a sheaf with a (G, H) -biaction. Then P is a (G, H) -bitorsor if and only if P is a right H -torsor and $G \rightarrow \underline{\text{Aut}}^H(P)$ is an isomorphism (or P is a left G -torsor and $H \rightarrow \underline{\text{Aut}}^G(P)$ is an isomorphism).*

Proof. We prove only the first claim. In particular we can assume that P is an H -torsor. Moreover, as the claim is local, we can also assume $P = H$. We therefore have a map

$$\phi: G \rightarrow \underline{\text{Aut}}^H(H) \simeq H \text{ such that } gh = \phi(g)h$$

As 1 is a global section of $P = H$, P is a G -torsor (that is P is a (G, H) -bitorsor) if and only if the orbit map $G \rightarrow H$, $g \mapsto g \cdot 1$ is an isomorphism. Since this last map is ϕ we get the claim. \square

We want to define a functor

$$\Lambda: \mathbf{B}(G, H) \rightarrow \underline{\mathbf{Hom}}(\mathbf{Sh}_{\mathcal{C}}^G, \mathbf{Sh}_{\mathcal{C}}^H)$$

As usual, we define this only over the global sections. If P is a (G, H) -torsor over \mathcal{C} and $X \in \mathbf{Sh}^G(\mathcal{C})$, then $X \times P$ has a right free action of G given by $(x, p)g = (xg, g^{-1}p)$. Its quotient $\Lambda_P(X) = (X \times P)/G$ has a right H -action induced by the one of P , so that $\Lambda_P(X) \in \mathbf{Sh}^H(\mathcal{C})$. Notice that, since G acts freely on $X \times P$, the quotient $(X \times P)/G$ can be defined avoiding to sheafify the naive quotient (and thus avoiding the corresponding set theoretic problems): $(X \times P)/G: \mathcal{C} \rightarrow (\mathbf{Sets})$ maps an object $S \in \mathcal{C}$ to the set of G -torsors $Q \rightarrow S$ together with a G -equivariant map $Q \rightarrow X \times P$, in other words the quotient stack $[X \times P/G]$ is actually equivalent to a sheaf.

Lemma A.5. *Let P be a (G, H) -torsor over $S \in \mathcal{C}$ with a section $p_0 \in P(S)$. Then:*

- 1) *the orbit maps $G_S \rightarrow P, g \mapsto gp_0$ and $H_S \rightarrow P, h \mapsto p_0h$ are isomorphisms and the induced map $\phi: G_S \rightarrow H_S$ is an isomorphism of groups such that $gp_0 = p_0\phi(g)$: in other words P is isomorphic to the (G, H) -torsor H with left G -action $gh = \phi(g)h$;*
- 2) *the composition $\mathbf{Sh}_{\mathcal{C}/S}^{G_S} \xrightarrow{\Lambda_P} \mathbf{Sh}_{\mathcal{C}/S}^{H_S} \rightarrow \mathbf{Sh}_{\mathcal{C}/S}^{G_S}$, where the second map is induced by ϕ , is isomorphic to the identity.*

Proof. Since P is a left G -torsor and right H -torsor the orbit maps are isomorphisms. Thus the map $\phi: G_S \rightarrow H_S$ is well defined: $\phi(g)$ is the unique element such that $gp_0 = p_0\phi(g)$. This also allows to prove that ϕ is an isomorphism of groups. For the second part, it is enough to notice that the maps

$$X \rightarrow \Lambda_P(X), x \mapsto (x, p_0), \quad \Lambda_P(X) \rightarrow X, (x, gp_0) \rightarrow xg$$

are well defined, inverses of each other and G -equivariant. \square

Remark A.6. Any (G, H) -baction on a sheaf P induces an (H, G) -baction on P by the rule: $g \star p \star h = h^{-1}pg^{-1}$. Moreover a (G, H) -torsor P is naturally also an (H, G) -torsor, which we denote by P^* , and this operation defines an isomorphism of stacks $\mathbf{B}(G, H) \rightarrow \mathbf{B}(H, G)$.

Lemma A.7. *If P is a (G, H) -torsor and $X \in \mathbf{Sh}^G(\mathcal{C})$ there is a canonical G -equivariant isomorphism of sheaves $\Lambda_{P^*}(\Lambda_P(X)) \rightarrow X$. In particular the functor $\Lambda_P: \mathbf{Sh}_{\mathcal{C}}^G \rightarrow \mathbf{Sh}_{\mathcal{C}}^H$ is an equivalence of stacks.*

Proof. There is a map $\psi: P \times P^* \rightarrow G$ determined by the rule $p = \psi(p, q)q$ for $p, q \in P$. A simple computation shows that $\psi(up, vq) = u\psi(p, q)v^{-1}$ for $u, v \in G$ and $\psi(ph, qh) = \psi(p, q)$ for $h \in H$. Using those expressions it is easy to show that the map

$$X \times P \times P^* \rightarrow X, (x, p, q) \mapsto x\psi(p, q)$$

factors through a G -equivariant map $\Lambda_{P^*}(\Lambda_P(X)) \rightarrow X$. Going locally where P has a section and using A.5 it is easy to show that this map is an isomorphism. \square

Proposition A.8. *The functor $\Lambda: \mathbf{B}(G, H) \rightarrow \underline{\mathbf{Hom}}(\mathbf{Sh}_{\mathcal{C}}^G, \mathbf{Sh}_{\mathcal{C}}^H)$ is an equivalence onto the full substack of functors which are fully faithful and maps G to an H -torsor. Moreover Λ_P is an equivalence for $P \in \mathbf{B}(G, H)$ and restricts to an equivalence $\Lambda_P: \mathbf{B}G \rightarrow \mathbf{B}H$. The induced functor $\Lambda: \mathbf{B}(G, H) \rightarrow \underline{\mathbf{Hom}}(\mathbf{B}G, \mathbf{B}H)$ is an equivalence onto the full substack of functors which are fully faithful.*

Proof. By A.5, we see that if $P \in \mathbf{B}(G, H)$ and $X \in \mathbf{B}G$ then $\Lambda_P(X) \in \mathbf{B}H$. Thus the functor $\Lambda: \mathbf{B}(G, H) \rightarrow \underline{\mathbf{Hom}}(\mathbf{B}G, \mathbf{B}H)$ is well defined and $\Lambda_P(G) \in \mathbf{B}H$. We prove both statement at the same time. For this set either $\underline{H} = \underline{\mathbf{Hom}}(\mathbf{Sh}_{\mathcal{C}}^G, \mathbf{Sh}_{\mathcal{C}}^H)$ or $\underline{H} = \underline{\mathbf{Hom}}(\mathbf{B}G, \mathbf{B}H)$ and denote by E the full substack considered in the statement.

By A.7 the functor $\Lambda: \mathbf{B}(G, H) \rightarrow \underline{H}$ has values in E . On the other hand if $\Omega \in E$ then $\Omega(G)$ is an H -torsor and the map

$$G \xrightarrow{\simeq} \underline{\mathbf{Aut}}^G(G) \rightarrow \underline{\mathbf{Aut}}^H(\Omega(G))$$

is an isomorphism because Ω is fully faithful. Thus $\Omega(G)$ is a (G, H) -bitorsor thanks to A.4. Evaluation in G therefore defines a functor $Y: E \rightarrow \mathbf{B}(G, H)$. We first show that $Y \circ \Lambda \simeq \text{id}_{\mathbf{B}(G, H)}$. The following maps

$$P \rightarrow \Lambda_P(G), p \mapsto (1, p), \quad \Lambda_P(G) \rightarrow P, (g, p) \mapsto gp \text{ for } P \in \mathbf{B}(G, H)$$

are quasi-inverses of each other and H -equivariant. Moreover we can check that they also preserve the G -action.

We now prove that $\Lambda \circ Y \simeq \text{id}_E$. Let $\Omega \in E$. Using the isomorphism $X \simeq \underline{\mathbf{Hom}}^G(G, X)$ we obtain natural morphisms

$$X \times \Omega(G) \simeq \underline{\mathbf{Hom}}^G(G, X) \times \Omega(G) \rightarrow \underline{\mathbf{Hom}}^H(\Omega(G), \Omega(X)) \times \Omega(G) \rightarrow \Omega(X), \quad (x, e) \mapsto \Omega(u_x)(e)$$

Here, for $(x, e) \in X(S) \times \Omega(G)(S)$ the map $u_x: G_S \rightarrow X_S$ is the orbit map $u_x(g) = xg$. By going through the definitions one can check that the above map is H -equivariant and G -invariant. Thus it induces a natural morphism $\Lambda_{\Omega(G)}(X) \rightarrow \Omega(X)$ that we claim is an isomorphism.

Since this is a local claim and $\Omega(G)$ is an H -torsor, we can assume that $\Omega(G)$ has a section. Taking into account A.5, we can assume $G = H$ and $\Omega(G) = H$ with left and right actions given by multiplication. We now prove that $\Omega \simeq \text{id}$, so that, in particular, the morphism $\Lambda_{\Omega(G)} \rightarrow \Omega$ will be an isomorphism. We have a natural isomorphism of sheaves of sets

$$\delta: X \simeq \underline{\mathbf{Hom}}^G(G, X) \simeq \underline{\mathbf{Hom}}^H(\Omega(G), \Omega(X)) = \underline{\mathbf{Hom}}^H(H, \Omega(X)) \simeq \Omega(X), \quad \delta(x) = \Omega(u_x)(1)$$

We just have to show that it is G -equivariant. On the other hand $u_{xg} = u_x \circ m_g$, where $m_g: G \rightarrow G$ is the left multiplication by g and, by construction, $\Omega(m_g)(1) = g \cdot 1 = g$, because the left action of G on $\Omega(G) = H$ is just the left multiplication for $G = H$. In particular

$$\delta(xg) = \Omega(u_{xg})(1) = \Omega(u_x)(\Omega(m_g)(1)) = \Omega(u_x)(g) = \Omega(u_x)(1 \cdot g) = \Omega(u_x)(1)g = \delta(x)g$$

as required. \square

Corollary A.9. *We have $\mathbf{B}G \simeq \mathbf{B}H$ if and only if there exists an H -torsor P over \mathcal{C} with an isomorphism $G \simeq \underline{\mathbf{Aut}}^H P$.*

We now want to describe two examples of non trivial bitorsors.

Example A.10. Set $P = \underline{\mathbf{Iso}}(G, H)$. The maps

$$\underline{\mathbf{Aut}} G \times P \longrightarrow P, \quad P \times \underline{\mathbf{Aut}}(H) \longrightarrow P, \text{ both given by } (\phi, \psi) \longmapsto \phi \circ \psi$$

induce a $(\underline{\mathbf{Aut}} G, \underline{\mathbf{Aut}} H)$ -action on $\underline{\mathbf{Iso}}(G, H)$ and, if G and H are locally isomorphic, then $\underline{\mathbf{Iso}}(G, H)$ is a $(\underline{\mathbf{Aut}} G, \underline{\mathbf{Aut}} H)$ -bitorsor. In particular, in this case, we obtain an isomorphism

$$\mathbf{B} \underline{\mathbf{Aut}}(G) \simeq \mathbf{B} \underline{\mathbf{Aut}}(H)$$

The second bitorsor we want to describe is a refinement of the previous one.

Proposition A.11. *Set $P = G \times \underline{\mathbf{Iso}}(H, G)$. The maps*

$$P \times (H \rtimes \underline{\mathbf{Aut}} H) \longrightarrow P, \quad (G \rtimes \underline{\mathbf{Aut}} G) \times P \longrightarrow P, \text{ both given by } (x, \phi) \cdot (y, \psi) = (x\phi(y), \phi\psi)$$

define a $((G \rtimes \underline{\mathbf{Aut}} G), (H \rtimes \underline{\mathbf{Aut}} H))$ -action on P and, if G and H are locally isomorphic, then P is a $((G \rtimes \underline{\mathbf{Aut}} G), (H \rtimes \underline{\mathbf{Aut}} H))$ -bitorsor. In particular, in this case, we have an isomorphism

$$\mathbf{B}(G \rtimes \underline{\mathbf{Aut}} G) \simeq \mathbf{B}(H \rtimes \underline{\mathbf{Aut}} H)$$

and, if $\Lambda_P: \mathrm{Sh}_{\mathcal{C}}^{G \rtimes \underline{\mathrm{Aut}} G} \rightarrow \mathrm{Sh}_{\mathcal{C}}^{H \rtimes \underline{\mathrm{Aut}} H}$ is the functor defined in A.8, we have a canonical isomorphism of sheaves of sets

$$(X/\underline{\mathrm{Aut}} G) \simeq (\Lambda_P(X)/\underline{\mathrm{Aut}} H) \text{ for all } X \in \mathrm{Sh}_{\mathcal{C}}^{G \rtimes \underline{\mathrm{Aut}} G}$$

Proof. A direct computation shows that the maps in the statement yield compatible actions. Moreover, if $\gamma: G \rightarrow H$ is an isomorphism, it is also straightforward to check that the maps $g \mapsto g \cdot \gamma$ and $h \mapsto \gamma \cdot h$ are equivariant isomorphisms $G \rtimes \underline{\mathrm{Aut}} G \rightarrow P$ and $H \rtimes \underline{\mathrm{Aut}} H \rightarrow P$ respectively. Finally consider the map

$$\pi: X \times P = X \times G \times \underline{\mathrm{Iso}}(H, G) \rightarrow X \text{ given by } \pi(x, g, \phi) = x(g, \mathrm{id}_G)$$

It is easy to check that $\pi(z(u, \psi)) = \pi(z)(1_G, \psi)$ and $\pi(z(1_H, \delta)) = \pi(z)$ for all $z \in X \times P$, $(u, \psi) \in G \rtimes \underline{\mathrm{Aut}} G$ and $(1_H, \delta) \in H \rtimes \underline{\mathrm{Aut}} H$. In particular π yields a map $(\Lambda_P(X)/\underline{\mathrm{Aut}} H) \rightarrow (X/\underline{\mathrm{Aut}} G)$. This is an isomorphism since it is so locally, i.e. when we have an isomorphism $H \xrightarrow{\phi} G$: in this case the inverse is given by $x \rightarrow (x, 1_G, \phi)$. \square

APPENDIX B. SHEAVES AND TRACES

In this appendix we want to collect several constructions involving quasi-coherent sheaves, especially locally free ones, that are used throughout the paper. In particular we will introduce and discuss the trace map associated with particular morphisms of sheaves. We fix a base scheme T . All sheaves will be defined over this scheme.

Remark B.1. If \mathcal{F} is a locally free sheaf of rank 2 then the canonical map $\mathcal{F} \otimes \mathcal{F} \rightarrow \det \mathcal{F}$ induces an isomorphism

$$\mathcal{F} \simeq \mathcal{F}^\vee \otimes \det \mathcal{F}$$

If y, z is a basis of \mathcal{F} then the above map is given by $y \rightarrow -z^* \otimes (y \wedge z)$, $z \rightarrow y^* \otimes (y \wedge z)$.

Definition B.2. Let \mathcal{F} be a locally free sheaf, \mathcal{Q} be an \mathcal{O}_T -module and $\zeta: \mathcal{Q} \otimes \mathcal{F} \rightarrow \mathcal{F}$ be a morphism. We define $\mathrm{tr} \zeta: \mathcal{Q} \rightarrow \mathcal{O}_T$, called the *trace* of ζ , as

$$\mathrm{tr} \zeta: \mathcal{Q} \rightarrow \underline{\mathrm{End}}(\mathcal{F}) \simeq \mathcal{F}^\vee \otimes \mathcal{F} \xrightarrow{\mathrm{ev}} \mathcal{O}_T$$

where the last map is the evaluation. We denote by $\underline{\mathrm{Hom}}_{\mathrm{tr}=0}(\mathcal{Q} \otimes \mathcal{F}, \mathcal{F})$ the subsheaf of morphisms $\underline{\mathrm{Hom}}(\mathcal{Q} \otimes \mathcal{F}, \mathcal{F})$ of maps whose trace is 0. In particular

$$\underline{\mathrm{Hom}}_{\mathrm{tr}=0}(\mathcal{Q} \otimes \mathcal{F}, \mathcal{F}) \simeq \underline{\mathrm{Hom}}(\mathcal{Q}, \underline{\mathrm{Hom}}_{\mathrm{tr}=0}(\mathcal{F}, \mathcal{F}))$$

If \mathcal{A} is a locally free sheaf of algebras on T then its trace map is $\mathrm{tr}_{\mathcal{A}} = \mathrm{tr}(\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}): \mathcal{A} \rightarrow \mathcal{O}_T$.

Remark B.3. If a basis of \mathcal{F} is fixed and $\mathrm{rk} \mathcal{F} = n$ then $\underline{\mathrm{End}}(\mathcal{F}) \simeq \mathcal{M}_{n,n}$, the quasi-coherent sheaf of $n \times n$ -matrices and the composition

$$\mathcal{M}_{n,n} \simeq \underline{\mathrm{End}}(\mathcal{F}) \simeq \mathcal{F}^\vee \otimes \mathcal{F} \xrightarrow{\mathrm{ev}} \mathcal{O}_T$$

is the usual trace of matrices. In particular if $\zeta: \mathcal{Q} \otimes \mathcal{F} \rightarrow \mathcal{F}$ is a map then $(\mathrm{tr} \zeta)(q) = \mathrm{tr}(\zeta(q \otimes -))$.

Definition B.4. Let \mathcal{F} be a locally free sheaf, \mathcal{Q} be an \mathcal{O}_T -module and $\zeta: \mathcal{Q} \otimes \mathcal{F} \rightarrow \mathcal{F}$ be a morphism. We say that ζ is locally a multiple of the identity if all the morphisms in the image of $\mathcal{Q} \rightarrow \underline{\mathrm{End}}(\mathcal{F})$ are fpqc locally a multiple of the identity. We denote by $\underline{\mathrm{Hom}}_{\lambda \mathrm{id}}(\mathcal{Q} \otimes \mathcal{F}, \mathcal{F})$ the subsheaf of $\underline{\mathrm{Hom}}(\mathcal{Q} \otimes \mathcal{F}, \mathcal{F})$ of those morphisms.

Lemma B.5. *Let \mathcal{F} be a locally free sheaf and \mathcal{Q} be an \mathcal{O}_T -module. Then $\mathcal{O}_T \simeq \underline{\mathrm{End}}_{\lambda \mathrm{id}}(\mathcal{F})$, that is a morphism $\mathcal{F} \rightarrow \mathcal{F}$ is locally a multiple of the identity if it is a multiple of the identity. More generally, applying $\underline{\mathrm{Hom}}(\mathcal{Q}, -)$, we obtain*

$$\underline{\mathrm{Hom}}(\mathcal{Q}, \mathcal{O}_T) \simeq \underline{\mathrm{Hom}}(\mathcal{Q}, \underline{\mathrm{End}}_{\lambda \mathrm{id}}(\mathcal{F})) = \underline{\mathrm{Hom}}_{\lambda \mathrm{id}}(\mathcal{Q} \otimes \mathcal{F}, \mathcal{F}), \mu \mapsto \mu \otimes \mathrm{id}$$

Proof. The map $\mathcal{O}_T \rightarrow \underline{\text{End}}_{\lambda\text{id}}(\mathcal{F})$ is injective. Moreover it is an epimorphism in the fpqc topology, hence it is an isomorphism. The last claim follows from definition. \square

Lemma B.6. *Let \mathcal{F} be a locally free sheaf and \mathcal{Q} be an \mathcal{O}_T -module. Then there is an exact sequence of \mathcal{O}_T -modules*

$$0 \rightarrow \underline{\text{Hom}}_{\text{tr}=0}(\mathcal{Q} \otimes \mathcal{F}, \mathcal{F}) \rightarrow \underline{\text{Hom}}(\mathcal{Q} \otimes \mathcal{F}, \mathcal{F}) \xrightarrow{\text{tr}} \underline{\text{Hom}}(\mathcal{Q}, \mathcal{O}_T) \rightarrow 0$$

If $\text{rk } \mathcal{F} \in \mathcal{O}_T^*$ then tr restricts to an isomorphism $\underline{\text{Hom}}_{\lambda\text{id}}(\mathcal{Q} \otimes \mathcal{F}, \mathcal{F}) \rightarrow \underline{\text{Hom}}(\mathcal{Q}, \mathcal{O}_T)$, whose inverse is

$$\underline{\text{Hom}}(\mathcal{Q}, \mathcal{O}_T) \rightarrow \underline{\text{Hom}}_{\lambda\text{id}}(\mathcal{Q} \otimes \mathcal{F}, \mathcal{F}), \mu \mapsto (\mu / \text{rk } \mathcal{F}) \otimes \text{id}_{\mathcal{F}}$$

In particular in this case $\underline{\text{Hom}}(\mathcal{Q} \otimes \mathcal{F}, \mathcal{F}) = \underline{\text{Hom}}_{\text{tr}=0}(\mathcal{Q} \otimes \mathcal{F}, \mathcal{F}) \oplus \underline{\text{Hom}}_{\lambda\text{id}}(\mathcal{Q} \otimes \mathcal{F}, \mathcal{F})$.

Proof. The surjectivity of tr can be checked locally when \mathcal{F} is free. The second claim instead follows easily from B.5. \square

Remark B.7. Let R be a ring, M, N be R -modules and $m \in N$. Given $\eta: M \otimes N \rightarrow R$ we define

$$\Lambda^m M \otimes \Lambda^m N \rightarrow R, (x_1 \wedge \cdots \wedge x_m) \otimes (y_1 \wedge \cdots \wedge y_m) \mapsto \det((\eta(x_i \otimes y_j)))$$

Applying this construction to the evaluation $M^\vee \otimes M \rightarrow R$ we obtain a map

$$\Lambda^m(M^\vee) \rightarrow (\Lambda^m M)^\vee, (\phi_1 \wedge \cdots \wedge \phi_m) \mapsto (y_1 \wedge \cdots \wedge y_m \mapsto \det(\phi_i(y_j)))$$

A direct check shows that if M is free with basis e_1, \dots, e_n then $(e_{i_1}^* \wedge \cdots \wedge e_{i_m}^*)$ is mapped to $(e_{i_1} \wedge \cdots \wedge e_{i_m})^*$ for $1 \leq i_1 < \cdots < i_m \leq n$. In particular the above map is an isomorphism for locally free R -modules.

The map $\Lambda^m M \otimes \Lambda^m N \rightarrow R$ can also be obtained as

$$M \otimes N \rightarrow R \implies M \rightarrow N^\vee \implies \Lambda^m M \rightarrow \Lambda^m(N^\vee) \rightarrow (\Lambda^m N)^\vee \implies \Lambda^m M \otimes \Lambda^m N \rightarrow R$$

Remark B.8. If \mathcal{A} is a locally free sheaf of algebras its discriminant $\Delta_{\mathcal{A}}: (\det \mathcal{A})^2 \rightarrow \mathcal{O}_T$ is the map induced by $\text{tr}_{\mathcal{A}}(- \cdot -): \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{O}_T$ as in B.7. It defines an effective Cartier divisor on T whose complement coincides with the étale locus of $\text{Spec } \mathcal{A} \rightarrow T$ (see [Gro71, Proposition 4.10]). Assume that $\mathcal{A} = \mathcal{O}_T \oplus \mathcal{E}$ with $(\text{tr}_{\mathcal{A}})_{|\mathcal{E}} = 0$ and denote by $\pi: \mathcal{A} \rightarrow \mathcal{O}_T$ the projection. The map $\mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{A} \xrightarrow{\pi} \mathcal{O}_T$ also defines a map $\Delta: ((\det \mathcal{A})^2 \simeq) (\det \mathcal{E})^2 \rightarrow \mathcal{O}_T$ via B.7 and $\Delta_{\mathcal{A}} = n^n \Delta$, because $\text{tr}_{\mathcal{A}} = \text{tr}_{\mathcal{A}} \circ \pi = \text{tr}_{\mathcal{A}}(1)\pi = n\pi$.

B.1. Trace zero maps of the form $\text{Sym}^2 \mathcal{F} \rightarrow \mathcal{F}$. In this subsection \mathcal{F} will denote a locally free sheaf of rank 2.

Notation B.9. Given $\beta: \text{Sym}^2 \mathcal{F} \rightarrow \mathcal{F}$ we set

$$\text{tr } \beta = \text{tr}(\mathcal{F} \otimes \mathcal{F} \rightarrow \text{Sym}^2 \mathcal{F} \xrightarrow{\beta} \mathcal{F}): \mathcal{F} \rightarrow \mathcal{O}_T$$

If y, z is a basis of \mathcal{F} and $\beta(y^2) = ay + bz$, $\beta(yz) = cy + dz$, $\beta(z^2) = ey + fz$, then $(\text{tr } \beta)(y) = a + d$, $(\text{tr } \beta)(z) = c + f$.

Proposition B.10. [Mir85, BV12] *If $\beta: \text{Sym}^2 \mathcal{F} \rightarrow \mathcal{F}$ is a map such that $\text{tr } \beta = 0$ then there exists a unique dashed map δ as in*

$$\begin{array}{ccc} \text{Sym}^2 \mathcal{F} \otimes \mathcal{F} & \xrightarrow{\beta \otimes \text{id}} & \mathcal{F} \otimes \mathcal{F} \\ \downarrow & & \downarrow \\ \text{Sym}^3 \mathcal{F} & \xrightarrow{\delta} & \det \mathcal{F} \end{array}$$

This association yields an isomorphism

$$(B.1) \quad \begin{array}{ccc} \underline{\mathrm{Hom}}_{\mathrm{tr}=0}(\mathrm{Sym}^2 \mathcal{F}, \mathcal{F}) & \longrightarrow & \underline{\mathrm{Hom}}(\mathrm{Sym}^3 \mathcal{F}, \det \mathcal{F}) \\ \left(\begin{array}{ccc} a & c & e \\ b & -a & -c \end{array} \right) & \longmapsto & \left(\begin{array}{ccc} -b & a & c \\ & & e \end{array} \right) \end{array}$$

where the last row describes how this map behaves if a basis y, z of \mathcal{F} is chosen. Here the chosen basis for $\mathrm{Sym}^2 \mathcal{F}$ and $\mathrm{Sym}^3 \mathcal{F}$ are y^2, yz, z^2 and y^3, y^2z, yz^2, z^3 respectively.

Notation B.11. In the hypothesis of B.10 we will denote the correspondence (B.1) by $\beta \mapsto \delta_\beta$ and $\delta \mapsto \beta_\delta$. Given $\delta: \mathrm{Sym}^3 \mathcal{F} \rightarrow \det \mathcal{F}$ we also define maps

$$\eta_\delta: \mathrm{Sym}^2 \mathcal{F} \rightarrow \mathcal{O}_T, \quad \alpha_\delta: \det \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F} \quad \text{and} \quad m_\delta: (\det \mathcal{F})^2 \rightarrow \mathcal{O}_T$$

as follows. Define η_δ as the map

$$(B.2) \quad \mathrm{Sym}^2 \mathcal{F} \xrightarrow{u} \Lambda^2 \mathrm{Sym}^2 \mathcal{F} \otimes \Lambda^2 \mathcal{F}^\vee \xrightarrow{v} \mathcal{O}_S$$

where v is induced by $\Lambda^2 \beta_\delta: \Lambda^2 \mathrm{Sym}^2 \mathcal{F} \rightarrow \Lambda^2 \mathcal{F}$ and u is induced by

$$\begin{array}{ccc} \Lambda^2 \mathcal{F} \otimes \mathrm{Sym}^2 \mathcal{F} & \longrightarrow & \Lambda^2 \mathrm{Sym}^2 \mathcal{F} \\ (x_1 \wedge x_2) \otimes x_3 x_4 & \longmapsto & -x_1 x_3 \wedge x_2 x_4 - x_1 x_4 \wedge x_2 x_3 \end{array}$$

If $2 \in \mathcal{O}_T^*$, the map α_δ

$$\det \mathcal{F} \otimes \mathcal{F}^\vee \otimes \mathcal{F} \xrightarrow{\simeq} \mathcal{F} \otimes \mathcal{F} \xrightarrow{\eta_\delta/2} \mathcal{O}_S$$

where $\det \mathcal{F} \otimes \mathcal{F}^\vee \simeq \mathcal{F}$ is the canonical isomorphism of B.1 and m_δ is

$$(\det \mathcal{F})^2 \otimes \det \mathcal{F} \simeq \det(\det \mathcal{F} \otimes \mathcal{F}) \xrightarrow{-\det \alpha_\delta} \det \mathcal{F}$$

Remark B.12. In the hypothesis of B.10, if y, z is a basis of \mathcal{F} , we identify $\det \mathcal{F} \simeq \mathcal{O}_T$ using the generator $y \wedge z \in \det \mathcal{F}$ and we write δ as

$$(B.3) \quad \delta(y^3) = -b, \quad \delta(y^2 z) = a, \quad \delta(y z^2) = c, \quad \delta(z^3) = e$$

then we have expressions

$$(B.4) \quad \begin{aligned} \eta_\delta(y^2) &= 2(a^2 + bc), \quad \eta_\delta(yz) = ac + be, \quad \eta_\delta(z^2) = 2(c^2 - ae) \\ 2\alpha_\delta(y) &= \eta_\delta(yz)y - \eta_\delta(y^2)z, \quad 2\alpha_\delta(z) = \eta_\delta(z^2)y - \eta_\delta(yz)z \end{aligned}$$

B.2. Trace zero maps of the form $\mathcal{Q} \otimes \mathcal{F} \rightarrow \mathcal{F}$. In this subsection \mathcal{F} will denote a locally free sheaf of rank 2 and \mathcal{Q} an \mathcal{O}_T -module. Moreover we assume $2 \in \mathcal{O}_T^*$.

Remark B.13. Given a map $\alpha: \mathcal{Q} \otimes \mathcal{F} \rightarrow \mathcal{F}$, we have a factorization

$$\begin{array}{ccc} s \otimes p \otimes q & \longmapsto & \alpha(s \otimes q)p - \alpha(s \otimes p)q \\ \mathcal{Q} \otimes \mathcal{F} \otimes \mathcal{F} & \longrightarrow & \mathrm{Sym}^2 \mathcal{F} \\ & \searrow & \nearrow \\ & \mathcal{Q} \otimes \det \mathcal{F} & \end{array}$$

and this association defines a map

$$(B.5) \quad \begin{array}{ccc} \underline{\mathrm{Hom}}(\mathcal{Q} \otimes \mathcal{F}, \mathcal{F}) & \longrightarrow & \underline{\mathrm{Hom}}(\mathcal{Q} \otimes \det \mathcal{F}, \mathrm{Sym}^2 \mathcal{F}) \\ \left(\begin{array}{cc} u & v \\ w & z \end{array} \right) & \longmapsto & \left(\begin{array}{ccc} v & z - u & -w \end{array} \right) \end{array}$$

where the last row describes the behaviour if $\mathcal{Q} = \mathcal{O}_T$ and a basis y, z of \mathcal{F} is given.

If $\mathcal{Q} = \mathcal{O}_T$ we obtain a map $\underline{\text{End}}(\mathcal{F}) \rightarrow \underline{\text{Hom}}(\det \mathcal{F}, \text{Sym}^2 \mathcal{F})$ and the general map (B.5) is obtained applying $\underline{\text{Hom}}(\mathcal{Q}, -)$ to the previous map.

Lemma B.14. *The map (B.5) restricts to an isomorphism*

$$\underline{\text{Hom}}_{\text{tr}=0}(\mathcal{Q} \otimes \mathcal{F}, \mathcal{F}) \rightarrow \underline{\text{Hom}}(\mathcal{Q} \otimes \det \mathcal{F}, \text{Sym}^2 \mathcal{F})$$

and its kernel is $\underline{\text{Hom}}_{\lambda \text{id}}(\mathcal{Q} \otimes \mathcal{F}, \mathcal{F}) \simeq \underline{\text{Hom}}(\mathcal{Q}, \mathcal{O}_T)$. In particular

$$\mathcal{O}_T \oplus \underline{\text{Hom}}(\det \mathcal{F}, \text{Sym}^2 \mathcal{F}) \simeq \underline{\text{End}}(\mathcal{F})$$

Proof. All the claims follows from the case $\mathcal{Q} = \mathcal{O}_T$, the local description of (B.5) and B.6. \square

Remark B.15. If \mathcal{L} is an invertible sheaf the map

$$\begin{aligned} \text{Sym}^2(\underline{\text{Hom}}(\mathcal{F}, \mathcal{L})) &\longrightarrow \underline{\text{Hom}}(\text{Sym}^2 \mathcal{F}, \mathcal{L}^2) \\ \xi \eta &\longmapsto (uv \longmapsto \xi(u)\eta(v) + \eta(u)\xi(v)) \end{aligned}$$

is an isomorphism: if $\mathcal{L} = \mathcal{O}_T$ and a basis y, z of \mathcal{F} is given, the above association maps $(y^*)^2, y^*z^*, (z^*)^2$ to $2(y^2)^*, (yz)^*, 2(z^2)^*$.

Using (B.1), the composition

$$\text{Sym}^2 \mathcal{F} \simeq \text{Sym}^2(\underline{\text{Hom}}(\mathcal{F}, \det \mathcal{F})) \simeq \underline{\text{Hom}}(\text{Sym}^2 \mathcal{F}, (\det \mathcal{F})^2)$$

yields an isomorphism

$$\begin{aligned} \underline{\text{Hom}}((\det \mathcal{F})^2, \text{Sym}^2 \mathcal{F}) &\longrightarrow \underline{\text{Hom}}(\text{Sym}^2 \mathcal{F}, \mathcal{O}_T) \\ \begin{pmatrix} u & v & w \end{pmatrix} &\longmapsto \begin{pmatrix} 2w & -v & 2u \end{pmatrix} \end{aligned} \tag{B.6}$$

where the last row describes its behaviour if a basis y, z of \mathcal{F} is given.

Notation B.16. If \mathcal{N} is an invertible sheaf, applying $\underline{\text{Hom}}(\mathcal{N}, -)$ to (B.6) we obtain an isomorphism

$$\underline{\text{Hom}}(\mathcal{N} \otimes (\det \mathcal{F})^2, \text{Sym}^2 \mathcal{F}) \rightarrow \underline{\text{Hom}}(\text{Sym}^2 \mathcal{F}, \mathcal{N}^{-1})$$

This association will be denoted by $\zeta \longmapsto \check{\zeta}$ and $\hat{\eta} \longleftarrow \eta$. Notice that the above map has the same local description of (B.6).

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